of diode voltage are in good agreement with the calculated orbits of typical single particles.

In summary, an ion source giving sufficient beam quality for self-confined ring formation has been developed. Axial energy has been extracted from the ring by two collective interactions, with return currents and with a resistive wall. Finally, the ring has been reflected from a weak mirror. Based upon these observations, the theoretical scaling of energy dissipation in a resistive wall,[9] and extensive computer simulations,[8] it is expected that a proton ring can be trapped if the injected proton current is increased by a factor of 3–4, the resistive wall is optimized, and the shape of mirror region is modified.

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Bounds on the Electromagnetic, Elastic, and Other Properties of Two-Component Composites

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A simple relation between three-point correlation functions is derived for two-component materials. This relation enables one to find concise expressions for the Beran, Molyneux, and McCoy bounds on transport and elastic coefficients in terms of the volume fraction, f_1, and two fundamental geometric parameters, ζ_1 and η_1. The parameter ζ_1 also determines bounds on the complex electrical permittivity. No simple interpretation of ζ_1 and η_1 has been found.

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This work is concerned with estimating the transport and elastic properties of a macroscopically homogeneous and isotropic composite, given the transport and elastic properties of the two components. Once a relation is found for any one transport coefficient it applies to any other transport coefficient. Indeed finding the effective electrical conductivity, heat conductivity, magnetic permeability, or diffusivity is mathematically analogous[1] to finding the effective electrical permittivity, ϵ_e, given the permittivities, ϵ_1 and ϵ_2 of the components.

The effective thermal expansion coefficient and consequently the specific heats of the composite can also be determined,[2,3] once we have found the effective bulk modulus, κ_e, and the effective shear modulus, μ_e, in terms of the bulk moduli, κ_1 and κ_2, and the shear moduli, μ_1 and μ_2, of the components. Furthermore[4] the work of Goodier[5] relates the effective viscosity of a composite (such as a suspension of particles in a fluid) to μ_e for a composite of incompressible materials (such as a suspension of particles in a solid matrix). Because of these relations, I focus on finding ϵ_e, κ_e, and μ_e, given ϵ_1, ϵ_2, κ_1, κ_2, μ_1, μ_2, and information about the structure of the material, including the volume fractions, f_1 and $f_2=1-f_1$, of the components.

Beran, Molyneux, and McCoy[6-8] have derived bounds on ϵ_e, κ_e, and μ_e which provide both an estimate of each quantity and the uncertainty associated with the estimate. They assume that the scale of inhomogeneities is much larger than atomic dimensions and that the composite is free of internal stress when no external stresses are present. They make no questionable assumptions regarding the geometry of the material. In fact the structure may be disordered and the components may be intermeshed so that neither can be identified as inclusion or matrix. Their work and other work on bounds, notably of Hashin and Shtrikman,[1,9] and Walpole,[10] is discussed in the reviews of Hale[11] and Watt, Davis, and O'Connell.[12]

One disadvantage of the bounds derived by Beran, Molyneux, and McCoy, apart from their complexity, is that the upper and lower bounds depend on different three-point correlation functions. For instance, the upper and lower bounds on κ_e depend[7] on the correlation functions $\langle\mu'(0)\times\kappa'(\vec{r})\kappa'(\vec{s})\rangle$ and $\langle\kappa'(\vec{r})\kappa'(\vec{s})/\mu(0)\rangle$, respectively. Here the angular brackets denote an ensemble average and κ' and μ' denote the deviation of the bulk modulus κ and shear modulus μ from their averages, $\langle\kappa\rangle = f_1\kappa_1 + f_2\kappa_2$ and $\langle\mu\rangle = f_1\mu_1 + f_2\mu_2$, i.e., for the endpoint of a vector $\vec{r}$ in component-1 material, $\kappa'(\vec{r}) = \kappa_1 - \langle\kappa\rangle = f_2(\kappa_1 - \kappa_2)$. Now noting that

$$1/\mu(0) = \langle 1/\mu\rangle - \mu'(0)/\mu_1\mu_2, \tag{1}$$

where $\mu(0)$ is the shear modulus at the origin, I deduce that

$$\langle\kappa'(\vec{r})\kappa'(\vec{s})/\mu(0)\rangle = \langle 1/\mu\rangle\langle\kappa'(\vec{r})\kappa'(\vec{s})\rangle - \langle\mu'(0)\kappa'(\vec{r})\kappa'(\vec{s})\rangle/\mu_1\mu_2. \tag{2}$$

Equation (2) and similar relations, together with results given in appendix B of Corson's work[13] (after some corrections) and an identity given in appendix B of Miller's work,[14] enable us to simplify the bounds derived by Beran, Molyneux, and McCoy. Considerable algebraic manipulation is required to simplify the McCoy bounds, and another identity, similar to Miller's result, is used. The simplified bounds, which depend on two geometric parameters, $\zeta_1 = 1 - \zeta_2$ and $\eta_1 = 1 - \eta_2$, are presented in Table I. It can be shown that

$$\zeta_1 = \lim_{\Delta\to 0} \frac{9}{2f_1f_2}\int_\Delta^\infty dr \int_\Delta^\infty ds \int_{-1}^{+1} du \frac{Q_1(r,s,u)}{rs} P_2(u), \tag{3}$$

$$\eta_1 = \frac{5\zeta_1}{21} + \lim_{\Delta\to 0} \frac{150}{7f_1f_2}\int_\Delta^\infty dr \int_\Delta^\infty ds \int_{-1}^{+1} du \frac{Q_1(r,s,u)}{rs} P_4(u), \tag{4}$$

TABLE I. The simplified bounds on the effective electrical permittivity ϵ_e, bulk modulus κ_e, and shear modulus μ_e of the two-component composites. We define $\langle a\rangle = a_1f_1 + a_2f_2$, $\langle a\rangle_\zeta = a_1\zeta_1 + a_2\zeta_2$, $\langle a\rangle_\eta = a_1\eta_1 + a_2\eta_2$, and $\langle\tilde{a}\rangle = a_2f_1 + a_1f_2$, where a represents any property. (The tilde in $\tilde{x}$ denotes the operation interchanging the subscripts 1 and 2 on x.) Here $f_1 = 1 - f_2$ is the volume fraction of component 1. Expressions for the geometric parameters $\zeta_1 = 1 - \zeta_2$ and $\eta_1 = 1 - \eta_2$ are given in the text. Also we define $\Xi \equiv [10\langle\kappa\rangle^2\langle 1/\kappa\rangle_\zeta + 5\langle\mu\rangle\langle 3\mu + 2\kappa\rangle\langle 1/\mu\rangle_\zeta + \langle 3\kappa + \mu\rangle^2\langle 1/\mu\rangle_\eta]/\langle 9\kappa + 8\mu\rangle^2$, $\Theta \equiv [10\langle\mu\rangle^2\langle\kappa\rangle_\zeta + 5\langle\mu\rangle\langle 3\mu + 2\kappa\rangle\langle\mu\rangle_\zeta + \langle 3\kappa + \mu\rangle^2\langle\mu\rangle_\eta]/\langle\kappa + 2\mu\rangle^2$.

Property	Lower Bound (ε_{eL}, κ_{eL}, μ_{eL})	Upper Bound (ε_{eU}, κ_{eU}, μ_{eU})
ε_e	$\left[\langle 1/\varepsilon\rangle - \dfrac{2f_1f_2(1/\varepsilon_1 - 1/\varepsilon_2)^2}{2\langle 1/\tilde{\varepsilon}\rangle + \langle 1/\varepsilon\rangle_\zeta}\right]^{-1}$	$\left[\langle\varepsilon\rangle - \dfrac{f_1f_2(\varepsilon_1 - \varepsilon_2)^2}{\langle\tilde{\varepsilon}\rangle + 2\langle\varepsilon\rangle_\zeta}\right]$
κ_e	$\left[\langle 1/\kappa\rangle - \dfrac{4f_1f_2(1/\kappa_1 - 1/\kappa_2)^2}{4\langle 1/\tilde{\kappa}\rangle + 3\langle 1/\mu\rangle_\zeta}\right]^{-1}$	$\left[\langle\kappa\rangle - \dfrac{3f_1f_2(\kappa_1 - \kappa_2)^2}{3\langle\tilde{\kappa}\rangle + 4\langle\mu\rangle_\zeta}\right]$
μ_e	$\left[\langle 1/\mu\rangle - \dfrac{f_1f_2(1/\mu_1 - 1/\mu_2)^2}{\langle 1/\tilde{\mu}\rangle + 6\,\Xi}\right]^{-1}$	$\left[\langle\mu\rangle - \dfrac{6f_1f_2(\mu_1 - \mu_2)^2}{6\langle\tilde{\mu}\rangle + \Theta}\right]$

where $P_2(u)$ and $P_4(u)$ are Legendre polynomials[15] and $Q_1(r,s,u)$ is the probability of a triangle, with two sides of length r and s at angle $\cos^{-1}(u)$, having all three vertices lie in component-1 material when placed randomly in the composite. The parameters ζ_1 and η_1 (and hence ζ_2 and η_2) lie in the interval $[0,1]$.

The bounds in Table I are very restrictive when the properties of the two components are similar, i.e., when $\delta_\epsilon = \epsilon_1 - \epsilon_2$, $\delta_\kappa = \kappa_1 - \kappa_2$, and $\delta_\mu = \mu_1 - \mu_2$ are small. To third order in δ_ϵ, δ_κ, and δ_μ we find that $\epsilon_{eL} = \epsilon_{eU}$, $\kappa_{eL} = \kappa_{eU}$, and $\mu_{eL} = \mu_{eU}$. Interestingly, to the same order of approximation, the well-known self-consistent approximations (also called symmetric effective-medium theories) of Bruggeman,[16] Landauer,[17] Budianski,[18] and Hill[19] correspond to a choice of parameters $\zeta_1 = \eta_1 = f_1$. To this same order of approximation, the Hashin-Shtrikman bounds[1,9] on ϵ_e, κ_e, and μ_e correspond to $\zeta_1 = \eta_1 = 0$ and to $\zeta_1 = \eta_1 = 1$.

The lower bounds, ϵ_{eL} and κ_{eL}, and the upper bounds, ϵ_{eU} and κ_{eU}, as given in Table I, are always an improvement on the appropriate Hashin-Shtrikman[1,9] and Walpole[10] (HSW) bounds. This is not surprising, as the HSW bounds do not include information about the value of ζ_1. At the extreme values of ζ_1, $\zeta_1 = 0$ and $\zeta_1 = 1$, we find that $\epsilon_{eL} = \epsilon_{eU}$ and $\kappa_{eL} = \kappa_{eU}$, each of which also equals one of the appropriate HSW bounds.

By contrast the lower bound on μ_e, $\mu_{eL}(\zeta_1,\eta_1)$ as in Table I [or the upper bound $\mu_{eU}(\zeta_1,\eta_1)$], is sometimes less restrictive than the appropriate HSW bounds. We point out that if the parameters ζ_1 and η_1 are both unknown then, since $0 \leq \zeta_1 \leq 1$ and $0 \leq \eta_1 \leq 1$, the following inequalities must hold:

$$\min\{\mu_{eL}(0,0),\ \mu_{eL}(0,1),\ \mu_{eL}(1,0),\ \mu_{eL}(1,1)\} \leq \mu_e \leq \max\{\mu_{eU}(0,0),\ \mu_{eU}(0,1),\ \mu_{eU}(1,0),\ \mu_{eU}(1,1)\}. \quad (5)$$

These bounds are new and incorporate the same information about the composite as the HSW bounds. If $\delta_\mu \delta_\kappa \geq 0$, then the bounds (5) are less restrictive than the HSW bounds. However if $\delta_\mu \delta_\kappa < 0$, then in many cases, such as if δ_μ and δ_κ are small, these bounds are more restrictive than the appropriate HSW bounds. The bounds (5) can be up to 5 times narrower than the HSW bounds.

For a symmetric cell material, as described by Miller,[14,20] the bounds ϵ_{eL}, ϵ_{eU}, κ_{eL}, and κ_{eU} coincide with the bounds which Miller derives by using the results of Beran and Molyneux.[6,7] In this case,

$$\zeta_1 = f_1 + \tfrac{1}{2}(f_2 - f_1)(9G - 1), \quad (6)$$

where G is a constant defined by Miller, which depends only on the shape of each cell in the material. It follows from Miller's work that $\zeta_1 = f_1$ for spherical cells, $\zeta_1 = f_2$ for platelike cells, and $\zeta_1 = (f_2 + 3f_1)/4$ for needlelke cells. Similarly we find that f_1, f_2, and $(f_2 + 5f_1)/6$ correspond to the values of η_1 for symmetric cell materials with cells that are, respectively, spherical, platelike, and needlelike.

For time-varying electric fields, as in electromagnetic radiation, the permittivities ϵ_1, ϵ_2, and ϵ_e are frequency dependent and complex, and so Beran's bounds[6] are no longer appropriate. In this case it is convenient to introduce a variable τ: $\tau = (\epsilon_1 + \epsilon_2)/(\epsilon_1 - \epsilon_2)$. Provided that the wavelength is much larger than the scale of inhomogeneities, we may treat ϵ_e as a function of ϵ_2 and τ which can be represented[21,22] in the following closed rational form:

$$\epsilon_e = \epsilon_2 \prod_{n=0}^{m} (\tau - \tau_n')/(\tau - \tau_n), \quad (7)$$

where the constants τ_n' and τ_n are real and satisfy several constraints.[22] It follows from (7) and the expression for ϵ_e to third order in δ_e that

$$\sum_n (\tau_n)^3 - (\tau_n')^3 = 2f_1 + 4f_1 f_2(4\zeta_1 - 3 + 2f_2)/3. \quad (8)$$

By using (8) and the method described in Ref. 22, Beran's bounds on ϵ_e can be generalized to complex ϵ_1 and ϵ_2.[23]

We find that ϵ_e is confined to a region Λ of the complex plane, bounded by two circular arcs, each joining $\epsilon_{eL}(\zeta_1)$ and $\epsilon_{eU}(\zeta_1)$. One arc would pass through $\epsilon_{eL}(0) = \epsilon_{eU}(0)$ when extended, the other extended arc would pass through $\epsilon_{eL}(1) = \epsilon_{eU}(1)$. The region Λ is more restrictive than the bounds, for complex ϵ, previously derived by Bergman[24] and by us.[22] It was proved[22] that ϵ_e is confined to a region Ω'' of the complex plane. Interestingly, the boundary of Ω'' corresponds to the locus of $\epsilon_{eL}(\zeta_1)$ and $\epsilon_{eU}(\zeta_1)$ as ζ_1 is varied between 0 and 1. When ϵ_1 and ϵ_2 are real the region Λ becomes an interval on the real axis and the endpoints coincide with Beran's bounds.

If additional information about the composite (such as higher-order correlation functions or values ϵ_e at other frequencies) is known, then bounds on ϵ_e which incorporate this information can similarly be found.[23] As progressively more information is known the bounds converge rapidly to the exact value of ϵ_e, unless the ratio ϵ_1/ϵ_2 is infinite, zero, or real and negative. At each stage ϵ_e is confined to a region of the complex plane bounded by two circular arcs.

For a given composite the probability function $Q_1(r,s,u)$ and hence the fundamental parameters ζ_1 and η_1 can be determined by using Corson's method[25] together with computer analysis of cross-sectional photographs. Once this is done for a range of composites with different structures, the physical significance of ζ_1 and η_1 should be elucidated. Alternatively[21] by measuring ϵ_e and other transport properties we can estimate ζ_1 and hence obtain improved bounds on κ_e and μ_e. The converse is also true.

The extension of this work to the properties of fiber-reinforced materials will be presented elsewhere. The author thanks G. H. Derrick and R. C. McPhedran for their suggestions which lead to the expressions (3) and (4) for ζ_1 and η_1 and D. B. Melrose for comments on the manuscript. This work was completed while the author held a Commonwealth Postgraduate Scholarship.

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