Field Theory and the Anderson Model for Disordered Electronic Systems

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An exact mapping is constructed of the site-diagonally disordered electron system onto a renormalizable field theory of interacting matrices. The matrix model that emerges is the same as for off-diagonal disorder, and it is therefore concluded that for diagonal disorder also there is a kind of local gauge invariance near the mobility edge.

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We consider the Anderson model of a disordered electronic system defined by the single-particle Hamiltonian

$$H=\sum_{r,r'}H_{rr'}|r\rangle\langle r'|,\quad H_{rr'}=\epsilon_r\delta_{rr'}+v_{rr'} \qquad (1)$$

on a d-dimensional lattice $|r\rangle$. The energies ϵ_r are taken as Gaussian-distributed random variables

$$P[\epsilon_r]=N^{-1}\exp\{-(1/2\gamma)\sum_r \epsilon_r{}^2\} \qquad (2)$$

and the real symmetric matrix $v_{rr'}$ is fixed and of short range. It is well known that the averaged Green's functions of this problem can be mapped onto spin-correlation functions of a Landau-Ginzburg model. This has led many authors[1] to suggest that the mobility edge, which in a disordered electronic system separates extended from localized states, can be described by the critical point of this Landau-Ginzburg model. However, all these attempts have been shown to fail,[2] mainly because they predict a long-range behavior of the one-particle Green's function which is hard to reconcile with its known analytic properties. On the other hand, Wegner[3] recently has introduced a model in which both ϵ_r and $v_{rr'}$ are treated as random variables. It has been shown that this problem leads to a matrix model[4] well suited for a field-theoretic analysis.[5] However, in Wegner's model the short-range nature of the averaged one-particle Green's function $G_E(r,r')$ is built in from the outset since the ensemble is invariant under local gauge transformations $|r\rangle\to\pm|r\rangle$.

Reconsidering the problem of pure diagonal disorder, we therefore are faced with the following questions: (1) Can we construct a matrix model in analogy to the work of Schäfer and Wegner?[4] (2) Is $G_E(r,r')$ of short range, thus justifying the use of the locally gauge-invariant models? (3) Can we give a simple argument explaining the failure of the spin models? In this Letter we present the main results of an analysis of these questions. The details will be given elsewhere.

We first recall the representation of the Green's functions $\langle(E_p-H)^{-1}\rangle,\langle(E_p-H)^{-1}(E_{p'}-H)^{-1}\rangle$, averaged over disorder, in terms of Gaussian integrals. We specialize our treatment to two energies ($p=1,2$),

$$E_p=E-\tfrac{1}{2}g_p\omega,\quad g_p=(-1)^p i,\quad \omega>0, \qquad (3)$$

referring to the advanced and retarded quantities.

Application of the standard replica trick, followed by an integration over the disorder field ϵ_r leads to a generating function[4,6] F

$$F=-\ln\int D[\epsilon]P[\epsilon]\int D[S]\exp\{-\tfrac{1}{2}\sum_{r,r'}\sum_{p=1}^{2}\sum_{a=1}^{n}g_pS_a{}^p(r)(E_p-H)_{rr'}S_a{}^p(r')\}, \qquad (4)$$

where the index a runs over n replicas for the spin fields. The averaged one-particle and multiparticle Green's functions follow as pair and multipoint spin correlations, defined in the usual way, in the special limit of zero ($n=0$) replicas. We notice that in this particular representation the convergence of the integrals is guaranteed by the imaginary part of E_p.

With use of Gaussian weight, Eq. (2), the integrals over ϵ_r in (4) are readily performed, yielding the

Lagrangian

$$L[S]=\tfrac{1}{2}\sum_{rr'}\sum_{a=1}^{n}\sum_{p=1}^{2}g_p S_a^p(r)[(E-\tfrac{1}{2}g_p\omega)\delta_{rr'}-v_{rr'}]S_a^p(r')-\tfrac{1}{8}\gamma\sum_r\{\sum_{p=1}^{2}\sum_{a=1}^{n}g_p[S_a^p(r)]^2\}^2. \quad (5)$$

Equation (5) closely resembles the Landau-Ginzburg-Wilson Hamiltonian used to describe critical phenomena. We want to point out here that this resemblance is misleading in so far as *any approach working directly with the spin fields is likely to violate the symmetries of the problem.*

Wegner was the first to stress[6] that the Anderson problem involves a continuous symmetry. For $\omega=0$ the Lagrangian (5) is invariant under transformations of the group $O(n,n)$, i.e., under all operations which leave $(S^{(1)})^2-(S^{(2)})^2$ invariant. The term $-\frac{1}{4}\omega\sum_p g_p{}^2(S^p)^2$ breaks this invariance, and the quantity conjugate to ω is easily identified as the density of states $\rho(E)$. For all energies inside the band the symmetry therefore is spontaneously broken by $\rho(E)\neq 0$, down to the level $O(n)\times O(n)$. Thus invariance with respect to operations mixing $S^{(1)}$ and $S^{(2)}$ is lost; rotational invariance in the subspaces $p=1$ or $p=2$ separately is not affected, however. To set up a perturbative treatment we now have to choose some mean-field approximation incorporating the broken symmetry. Clearly, in this approximation at least one of the vectors $S^{(p)}$ must be nonzero, and this immediately leads to a breaking of the $O(n)$ symmetry in subspace p which should be left intact in a correct treatment. Within the spin formulation no mean-field approximation incorporates the correct symmetries. A more detailed analysis shows that as a result unstable modes are present which want to restore the $O(n)$ symmetry. This explains the failure of the spin models.

Following Ref. 4, we now transform to a matrix model. We decouple the quartic term $\sum_{p,p'}\sum_{a,a'}(G_p{}^{1/2}g_{p'}{}^{1/2}S_a^pS_{a'}^{p'})^2$ in $L[S]$ by a Gaussian transformation introducing a matrix field $Q_{aa'}{}^{pp'}(r)$. Integrating over the spin fields we find the effective Lagrangian

$$L[Q]=(2\gamma)^{-1}\mathrm{Tr}Q^2(r)-(\omega/2\gamma)\mathrm{Tr}gQ(r)+\tfrac{1}{2}\mathrm{Tr}\ln\{[E-Q(r)]\delta_{rr'}-v_{rr'}\}, \quad (6)$$

where the trace includes summation over p, a, r, and E,v are unit matrices in p,a. The matrix g is defined as $g_{aa'}{}^{pp'}=\delta_{pp'}\delta_{aa'}g_p$. The main problem that one faces here is to specify a domain for the Q matrices such that convergency and symmetry requirements are fulfilled. As a general remark, we can see from symmetry that the integration over composite variables $Q(r)$ should be invariant under the transformation $Q(r)\to g^{1/2}TQ(r)T^{-1}(g^\dagger)^{1/2}$, where T ($\equiv T_{aa'}{}^{pp'}$) is an element of the group $O(n,n)$. Furthermore, since Q replaces the $n(2n+1)$ different components of $g_p{}^{1/2}S_a^pS_{a'}^{p'}g_{p'}{}^{1/2}$, we get the correct number of independent components of Q by writing

$$Q(r)\to g^{1/2}T(r)P(r)T^{-1}(r)(g^\dagger)^{1/2}, \quad (7)$$

with both $T(r)$ and $P(r)$ real symmetric and in addition $P(r)=\delta_{pp'}P_{aa'}{}^p(r)$, $T(r)\in O(n,n)$. The particular representation of Eq. (7) is suggested by the saddle point equation of $L[Q]$, from which one concludes that the "longitudinal" components $P(r)$ are massive, whereas the $T(r)$ are massless for $\omega=0$. For a certain range of energies E and for nonzero ω, the saddle point can be written as

$$Q_{\mathrm{s.p.}}=\delta_{pp'}\delta_{aa'}(g_p\rho_0+e_0),\quad e_0 \text{ real},\quad \rho_0>0. \quad (8)$$

The real (e_0) and imaginary part (ρ_0) are determined by a coherent-potential-approximation-type equation which is the equivalent of the $n=\infty$ limit of Wegner's[3] n-orbital model. The $Q_{\mathrm{s.p.}}$ obviously fulfills the desired symmetry properties: It is invariant under $O(n)\times O(n)$ but not under $O(n,n)/[O(n)\times O(n)]$, thus providing a valid starting point for perturbation expansion, analogous to that of Oppermann and Wegner.[7] Equation (7) will now be employed by shifting

$$P(r)\to Q_{\mathrm{s.p.}}+P(r). \quad (9)$$

We stress that with the choice (7)–(9) the derivation of $L[Q]$ has been carried out in such a way that all integrals are manifestly convergent.

Having identified the massive fields $P(r)$ and massless ones $T(r)$, we next seek for an effective Lagrangian in $T(r)$, which formally follows after integration over $P(r)$. Using (7)–(9), we can write for the interaction term in $L[Q]$,

$$\tfrac{1}{2}\mathrm{Tr}\ln\{(E-e_0-g\rho_0)\delta_{rr'}-v_{rr'}-P(r)-T^{-1}(r)v_{rr'}[T(r')-T(r)]\}, \quad (10)$$

which then is expanded with respect to the last part of the argument. This expansion has the virtue of proceeding according to powers of $T(r')-T(r)=(r-r')_\mu\partial_\mu T(r)+\cdots$ and thus in the critical region near the mobility edge only terms up to second order need to be kept. However, it has the drawback of violating a local gauge invariance of the model. The model (6)–(9) is rigorously invariant under the transformation $T(r) \to T(r)O(r)$, where $O(r)$ is an element of $O(n)\times O(n)$. This invariance is no longer manifest in the expansion. However, analyzing the coefficient of the $(\partial_\mu)^2$ term of the effective Lagrangian for the $T(r)$ fluctuations we can show rigorously that the gauge-invariance–violating terms cancel by virtue of a special sum rule of the disordered electron problem. The effective Lagrangian then can be written in the form of the nonlinear σ model,

$$\tilde{L}[\tilde{O}] = \eta \operatorname{Tr}(\partial_\mu \tilde{O} \partial_\mu \tilde{O}), \tag{11}$$

where

$$O(r) = T(r) g T^{-1}(r). \tag{11a}$$

For the coefficient η we find a rigorous but complicated expression of (noncritical) correlation functions of the P fields. Evaluating this expression in saddle point approximation the form

$$\eta = \frac{\rho_0^2}{8d} \sum_r r^2 |\langle 0| (E - e_0 - i\rho_0 - v_{rr'})^{-1} |r\rangle|^2 \tag{12}$$

results, which is a coherent-potential-approximation–type expression for the conductivity.

Since $\tilde{L}$ is of the same structure as the nonlinear σ model derived in Ref. 4 for a case in which both ϵ_r and $v_{rr'}$ are random, we conclude that the diagonal-disorder problem and the off-diagonal-disorder problem with real symmetric $v_{rr'}$ (real matrix model) should show the same properties near the mobility edge. We may interpret this result as a proof that the diagonally disordered Anderson model in the asymptotic scaling region possesses the invariance with respect to local gauge transformations $|r\rangle \to \pm |r\rangle$, which is an essential feature of Wegner's models.

Calculating in our formalism the averaged one-particle Green's function, we find that the range of this function stays finite order for order in perturbation theory. Thus for the critical (i.e., long-range) behavior the gauge-invariant models are valid.

After completion of this work, we came to know a recent preprint of McKane and Stone[8] in which they also conclude that the diagonal-disorder model leads to the Lagrangians (5) and (11). Their argument, however, is purely formal insofar as they neither specify the set of matrices nor present a formulation free of convergence problems.

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