and $W_{ab} = |\sum_{i=1}^{N} V_a{}^i V_b{}^{i*}|^2 \geq 0$. Thus, $e(\rho_N) = E(K) - D$, with $2D = \mathrm{Tr}(L_2 v)$. But clearly L_2 is positive semidefinite, so that $D \geq 0$. This proves (i). To prove (ii), note that $E(K) \geq e(\rho_N) = \langle G^\theta \rangle_\theta$, where $G^\theta = \langle \psi_N{}^\theta, H_N \psi_N{}^\theta \rangle$ is real for each θ. Hence, for some θ, $G^\theta \leq e(\rho_N)$. Q.E.D.

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Note added.—After reading this manuscript, Professor M. B. Ruskai kindly pointed out that the lemma is essentially a consequence of Horn's theorem[1]: Let $y_1 \geq y_2 \geq \cdots \geq y_M$ and $x_1 \geq x_2 \geq \cdots \geq x_M$ be two sets of reals. Then there exists an $M \times M$ hermitean matrix B with eigenvalues $\{x_i\}$ and diagonal elements $B_{ii} = y_i$ if and only if $\sum_{i=1}^{t}(x_i - y_i) \geq 0$ for all $1 \leq t \leq M$, and with equality for $t = M$. The existence of B is equivalent to $y_j = \sum_{i=1}^{M} |U_{ij}|^2 x_i$ for some unitary U. To apply this to the lemma, suppose that $c_j = 0$ for $j > M \geq N$ and take $y_j = c_j$ (for $j \leq M$) and $x_1 = x_2 = \cdots = x_N = 1$, and $x_j = 0$ for $j > N$. The required orthonormal vectors V^i are then $V_j{}^i = U_{ij}$ for $j \leq M$ and $V_j{}^i = 0$ for $j > M$. Finally, if $c_j > 0$ for all j, then an argument such as that given at the end of the proof of the lemma, or something similar, must be used.

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Validity of Scaling to 10^{20} eV and High-Energy Cosmic-Ray Composition

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It is shown that evidence on cosmic-ray showers of energy 3×10^{16} to 10^{20} eV indicates that scaling in the fragmentation region is valid up to the highest energies if (and only if) hadron-air inelastic cross sections continue to rise in the manner observed at lower energies. It is also shown, with use of additional air-shower evidence, that $\langle \ln A \rangle$, the logarithmic mean primary mass number, changes from (4 ± 2) at 1.6×10^{15} eV to $(0^{+0.6}_{-0})$ at and above 3×10^{16} eV.

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Information about some features of nuclear interactions beyond 10^{12} eV can be obtained by the study of high-energy cosmic rays. Beyond 10^{14} eV these studies depend on observations of extensive air showers. Such observations, while not suited for the study of details, are capable of giving information about broad features. In particular, they can be used to test the validity of scaling in the fragmentation region. In this Letter we examine data on the depth of maximum development (X_m) of large air showers as a function of energy (E). The variation of X_m is related to the multiplicity law for the production of high-energy secondaries by the elongation-rate (ER) theorem.[1] By using this relation we show that one of the important predictions of scaling, namely that the multiplicity of high-energy secondaries is asymptotically energy independent,[2] is supported by air-shower evidence up to the highest observed energies, provided that hadron-air interaction cross sections continue to rise in the manner observed at lower energies.

In our analysis we have intentionally disregarded measurements of X_m by Thornton and Clay,[3] as their data have been challenged by Orford and Turver[4] on a number of grounds. We find, however, that the remaining evidence supports their conclusion as to a change in primary composition from heavy to light nuclei between $\sim10^{15}$ and 3×10^{16} eV.[3] This conclusion is especially interesting astrophysically because it is well established that the cosmic-ray spectrum between 2×10^{15} and 10^{17} eV is significantly steeper than at lower energies,[5] and there is evidence from a variety of experiments that the amplitude of cosmic-ray anisotropy increases rapidly with energy in the same region.[6]

We discuss the data on X_m in terms of D_e, the so-called "elongation rate," equal by definition to $dX_m/d\ln E$. X_m is averaged over fluctuations in shower development, and in case of mixed primary composition over the equal-energy mass spectrum. For numerical results we use "ER per decade," defined similarly in terms of $\log_{10}E$

and denoted by D_{10}. We make use of the ER theorem,[1] which states, for showers initiated by protons, that when D_e is expressed in the following form,

$$D_e = (1 - B)X_0 \quad (1)$$

(X_0 being the radiation length for electromagnetic cascades in air, equal to 37.7 g cm^{-2}), the quantity B, which expresses the dependence of D_e on hadron-air nucleus interactions, is nonnegative. Furthermore, B can be approximated by a series of simple terms, one of which can be written $B_g = d\ln g/d\ln E$, where g is the "effective multiplicity."[7] Here, following Gaisser *et al.*,[8] we add a term that takes into account rising cross sections, $B_\lambda = -\beta d(\lambda_N + \lambda_\pi)/d\ln E$, where βX_0 is a dimensionless constant of order unity, obtaining

$$B \simeq B_g + B_\lambda. \quad (2)$$

For scaling models $B_g = 0$. (For an extreme high-multiplicity model, $g \propto E^{1/2}$, it would equal 0.5.) For energies up to 10^{12} eV, the value of B is estimated to be 0.23 ± 0.12 on the basis of accelerator results.[8]

We also make use of the superposition principle, which states that an average shower produced by a nucleus with energy E and mass number A is indistinguishable, except in early stages of development, from a superposition of A average proton-initiated showers, each with energy E/A. It follows from this principle that Eq. (1) holds for primaries with any value of A, or for mixed composition if the primary mass spectrum is independent of energy. However, it must be modified in an energy interval where the composition is changing.

This is done most easily by assuming that B is not also changing in that interval. Equation (1) can then be integrated, after which superposition gives the result

$$X_m = X_1 + (1 - B)X_0(\ln E - \ln A), \quad (3)$$

where X_1 is model dependent but does not depend on E or A. Equation (3) states that the variation of X_m with $\ln E$ for various primary components is described by a family of parallel straight lines. It follows that for a mixed primary composition

$$\langle X_m \rangle = X_1 + (1 - B)X_0(\ln E - \langle \ln A \rangle), \quad (4)$$

where the angular brackets indicate averages over the mass spectrum as well as shower-development fluctuations. For a variable composition we can also write

$$D_e = (1 - B)X_0(1 - d\langle \ln A \rangle / d\ln E). \quad (5)$$

The estimate of B given above implies $D_{10} = 67 \pm 10$ g cm^{-2} for protons up to $\sim 10^{12}$ eV. Taking that as a reference, we see from Eq. (5) that the experimental value of D_{10} can differ from it because of either a change in hadronic interactions or a change in primary composition. We note that B cannot decrease very much. An increase in B_g to 0.25 with no attendant change in B_λ, which would imply a significant departure from scaling, would only change D_{10} from 67 to 45 g cm^{-2}. On the other hand, changes in D_{10} due to changes in primary composition could be quite dramatic. Taking as a reference the source-region charge composition derived by Rasmussen from direct measurements at low energies,[9] a change to pure Fe spread over a three-decade energy interval would reduce D_{10} from 67 to 42 g cm^{-2} throughout the interval. A change from pure Fe to pure H in a single decade would give $D_{10} = 184$ g cm^{-2}.

We first discuss in this context the X_m values listed in Table I and plotted in Fig. 1. The results of Kalmykov *et al.* are derived from measurements of the temporal distribution of atmospheric Čerenkov photons.[10] The errors quoted have been estimated by us from Eq. (6) of Ref. 10 and from a knowledge of the number of showers used. The results of Protheroe and Turver are obtained by a similar method.[11] In these experiments the derivation of X_m depends mainly on the well-understood theory of electromagnetic cascades. The highest-energy results of Table I are based on measurements of the temporal distribution of particles in showers of energy 3×10^{17}

TABLE I. Depth of shower maximum above 3×10^{16} eV.

Reference[a]	Energy (eV)	X_m (g cm^{-2})
Y(10)	3×10^{16}	675 ± 15
Y(10)	6.7×10^{16}	675 ± 15
D(11)	2×10^{17}	681 ± 20
Y(10)	5.6×10^{17}	712 ± 20
D(11)	2×10^{18}	766 ± 30
L(12)	10^{19}	805 ± 30[b]
L(12)	10^{20}	845 ± 80[b]

[a]Letters in column 1 refer to Fig. 1.

[b]Data of Ref. 12 have been normalized to 730 g cm^{-2} at 10^{18} eV (see text).

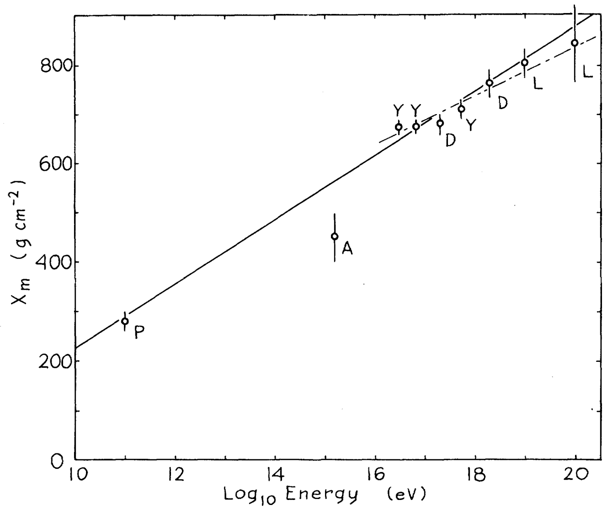

FIG. 1. X_m(g cm^{-2}) as a function of energy. The points above 10^{16} eV are identified in Table I. Point P, see text and Ref. 14. Point A, see text and Ref. 16. Derivation of the solid and dotted lines is described in the text.

to 10^{20} eV.[12] The calculated relation between X_m and the measured quantity is somewhat dependent in this case on the character of hadronic interactions at very high energies. Consequently, for the present purpose, these values of X_m have been normalized to 730 g cm^{-2} at 10^{18} eV so as to agree with Refs. 10 and 11, and the directly measured elongation rate[12] has been used to deduce the values of X_m quoted at 10^{19} and 10^{20} eV.

A weighted least-squares fit to these data gives $X_m(\text{g cm}^{-2}) = (290 \pm 80) + (50 \pm 10) \log_{10}E(\text{GeV})$. The fact that the data can be fitted by a straight line is itself a significant constraint on theories of the origin of these cosmic rays. It indicates that the well-known flattening of the energy spectrum above 10^{19} eV, which appears to have associated with it a change in the anisotropy,[13] is not accompanied by any striking change in primary composition. The fact that the experimental value of D_{10} agrees as well as it does with the reference value is an additional constraint, favoring models in which the primary composition does not change over the entire range from 3×10^{16} to 10^{20} eV.

We are allowed, therefore, to adopt the hypothesis that all of the primary particles above 3×10^{16} eV are protons, and we do so. Under it we can combine the data of Table I with lower-energy accelerator-based results in order to find X_1 and the average value of B over energies extending upward to 10^{20} eV. As no appropriate results have been published for air, we adopt those of Jones for nuclear cascades in water, an airlike medium.[14] Jones's results have been verified by a number of experiments with use of accelerators and cosmic rays, and they are consistent with the ER theorem from 10^{11} to 10^{13} eV. For our purpose we modify his calculations to include fluctuations in the position of the first interaction. Taking the proton mean free path in air to be 84.4 g cm^{-2} at 10^{11} eV, we find the value of X_m at that energy to be 280 ± 20 g cm^{-2}, where the error reflects approximations made in our adaptation of Jones's work.

A least-squares fit of this point and those used previously gives $X_m(\text{g cm}^{-2}) = (159 \pm 25) + (65 \pm 3) \times \log_{10}E(\text{GeV})$, with reduced $\chi^2 = 0.96$. The close agreement between the new value of D_{10} and the reference value means that the evidence given in Table I, together with Jones's result, supports the validity of scaling all the way up to 10^{20} eV, provided that hadron-air inelastic cross sections continue to rise in the manner observed at lower energies (provided, in other words, that $\overline{B}_\lambda \simeq 0.23$, where the bar designates an average over energies extending upward to 10^{20} eV). But if these cross sections behave otherwise (if, for instance, they decrease with energy at high energies), then scaling will have to be violated in the fragmentation region. The hypothesis that the primaries above 3×10^{16} eV are protons is not needed to draw these conclusions, because, according to Eq. (4), if those primaries were heavier than protons then the new value of D_{10} would imply $B < 0.23$, thus strengthening our argument.

We now turn to the question of the primary mass composition above 10^{15} eV. Aside from the previously mentioned work by Thornton and Clay,[3] only one experiment has given results comparable to those of Table I at a lower energy. They were obtained in a series of experiments by Antonov *et al.* in which the altitude variation of the shower rate from sea level to ~200 g cm^{-2} was measured.[15] The most precise data were obtained in two balloon flights of a 30-m-diam array of scintillation counters and Geiger-Müller counter hodoscopes. Antonov *et al.* give a detailed discussion of their experimental method and of the analysis techniques, and we can find no basis for the opinion[4] "that errors in assigned depths of maximum are at least 100 g cm^{-2}." Our own analysis of the balloon-flight data gives $X_m = 450 \pm 50$ g cm^{-2} for an energy of 1.6×10^{15} eV. This point is shown with the others in Fig. 1.

Our earlier result, that the value of B is independent of energy, allows us to use Eq. (4) for investigating the primary composition as measured by $\langle \ln A \rangle$. It is qualitatively clear from Fig. 1 that no single choice of $\langle \ln A \rangle$ will give a good fit to all of the data. However, it appears that a good fit can be obtained by assigning $\langle \ln A \rangle$ one value, $\langle \ln A_1 \rangle$, at 1.6×10^{15} eV and a different value, $\langle \ln A_2 \rangle$, at and above 3×10^{16} eV. (The point at 10^{11} eV is of course to be assigned the value $\langle \ln A \rangle = 0$.) In order to describe this situation quantitatively we have tested a range of choices of $\langle \ln A_1 \rangle$ and $\langle \ln A_2 \rangle$ by calculating χ^2 for the best fit of Eq. (4) to all of the data in Fig. 1, allowing the values of B and X_1 to vary freely.

The results are as follows:

(1) The best-fitting values of $\langle \ln A_1 \rangle$ and $\langle \ln A_2 \rangle$ are 4 ± 2 and $0_{-0}^{+0.6}$, respectively (standard errors, $\chi^2 = 1.2$). The best value of $\langle \ln A_1 \rangle$ corresponds to a pure-iron primary composition; the best value of $\langle \ln A_2 \rangle$, to pure protons. (For comparison, the reference charge composition[9] gives $\langle \ln A \rangle = 1.5$.) The best-fit value of B (0.25 ± 0.01) is reasonable. The 2-standard-deviation (2σ) upper limit of $\langle \ln A_2 \rangle$ is 1.1. If we interpret the mass distribution as comprised of two components, one having low-energy composition ($\langle \ln A \rangle = 1.5$) and the other made up of protons,[16] then the corresponding 2σ lower limit to the fraction of protons is 58%. (The 1σ limit in this case is 77%.)

(2) Adopting the hypothesis that $\langle \ln A \rangle = 1.5$ at 1.6×10^{15} eV, the 1σ and 2σ limits to the fraction of protons at and above 3×10^{16} eV are 83% and 72%, respectively. The value of χ^2 (1.6) is greater but not very improbable. The best-fit value of B (0.24 ± 0.01) is hardly changed.

(3) The assumption that $\langle \ln A_1 \rangle = \langle \ln A_2 \rangle = 1.5$ (no change in composition from low energies to 10^{20} eV) gives a decidedly poorer fit ($\chi^2 = 2.5$, with probability 0.02).

(4) The assumption that $\langle \ln A_1 \rangle = \langle \ln A_2 \rangle = 4$ (pure Fe at and above 1.6×10^{15} eV) seems to be excluded by the still poorer fit ($\chi^2 = 4.1$, with probability $\sim 10^{-3}$) and by the small value of B it would require (0.011 ± 0.001).

To summarize, the strongest conclusion from our analysis is that scaling, in the fragmentation region, appears to be valid up to 10^{20} eV if (and only if) hadron-air inelastic cross sections continue to rise in the manner observed at lower energies. This result is based on well-established measurements of the rate of change of depth of shower maximum from 3×10^{16} to 10^{20} eV and on a calculated value of the depth of maximum at 10^{11} eV, where the important features of hadronic interactions are well understood.

Our statements about the mass composition of cosmic rays are less strongly founded insofar as they rely rather heavily on the one point at 1.6×10^{15} eV by Antonov *et al.* We note that a strongly enhanced iron abundance at $\sim 10^{15}$ eV is indicated by delayed air-shower hadron measurements of Goodman *et al.*[17] There is a clear need for more work on the measurement of X_m, particularly between 10^{13} and 10^{16} eV.

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Note added.—Since this Letter was written, Thornton and Clay have published revised values of X_m.[18] Their new values are in good agreement with the results shown here in Fig. 1, both near 10^{15} eV and near 3×10^{16} eV. They strengthen the case presented in Ref. 3 and here for a change of composition.

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[7]The "effective multiplicity" in an inelastic hadron-nucleus collision is the multiplicity of secondaries that receive the bulk of the energy, excluding those with such relatively low values of laboratory energy that they have no effect on broad features of shower development. In terms of the energy distribution of secondary hadrons, φdE, it is the number of secondaries having energies greater than E_c, where E_c is defined by the condition $\int_{E_c}^{E_{\max}} \varphi \, dE = (1-\alpha) \int_0^{E_{\max}} \varphi \, dE$. The fraction α that characterizes "ineffective" secondaries depends on the state of the art of air-shower measurements. For current techniques we estimate $\alpha = 0.02$ to 0.05.

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Proton Decay Rate

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The pionic two-body decay amplitude of the proton in the SU(5) grand unified gauge theory is computed by using the soft-pion method in the reference frame of the pion at rest. The normalization condition of the relativistic three-body Bethe-Salpeter wave function of the proton is used in the computation. It is shown that the partial decay rate of the process, proton $\to e^+\pi^0$, is $(0.86\times10^{30}\text{ yr})^{-1}$ and $(1.4\times10^{31}\text{ yr})^{-1}$ for $m_X=4.0\times10^{14}$ GeV and 8.0×10^{14} GeV, respectively. These values are on the boundary of the present experimental limit.

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One of the most remarkable consequences of grand unified gauge theories is that the proton decays. Several authors have made estimates of the proton decay rate in the SU(5) and SO(10) models[1,2] and have found the lifetime to be in the range of $10^{31}-10^{33}$ yr,[3,4] which is close to the present experimental lower bound[5] ($\sim 10^{30}$ yr). Renewed attempts to observe such decays are under way and are expected to give some results soon if the proton lifetime is in the above-mentioned range.

Among various decay modes, the mode $p\to e^+ +\pi^0$ seems to be the most appropriate one for detection in most experiments in progress if its branching ratio is significantly large. However, there is some uncertainty in the theoretical estimates for this process since the estimates are made based on either the SU(6) wave functions or the Massachusetts Institute of Technology bag model wave functions, which are either nonrelativistic or noncovariant. The proton, consisting of three light quarks, is likely to be a relativistic system and the relativistic corrections may not be negligible. In this article, I present a relativistic calculation of the pionic two-body decay of protons based on the partially conserved axial-vector current hypothesis (PCAC) and current algebra.

In order to use the soft-pion method, I shall calculate the decay amplitude in the rest frame of the pion. If the soft-pion limit is taken literally, the momentum of the incident proton becomes infinite:

$$(p^2+M^2)^{1/2}=\frac{M^2+m^2+\mu_\pi^2}{2\mu_\pi}\to\infty,\ \text{as } \mu_\pi\to 0, \qquad (1)$$

where M, m, and μ_π are the masses of the proton, the positron, and the pion, respectively. However, it is important to note that we can use the soft-pion method either in the ordinary reference frame of the pion at rest or in the infinite-momentum reference frame of the incident proton.[6] The two situations are identical in the limit $\mu_\pi\to 0$. In fact, this method has been used in the analysis of nonleptonic hyperon weak decays and has led to successful sum rules for the p-wave amplitudes as well as for the s-wave amplitudes.[7]

The interaction Lagrangian in the SU(5) gauge model, which is relevant to the proton decay, is