Exact, Fractionally Charged Self-Dual Solution

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A finite-action self-dual solution of SU(2) gauge theory with topological charge $\frac{3}{2}$ is given.

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In this paper we give a new finite-action solution of the self-duality equations (SDE) with Pontryagin number $\frac{3}{2}$. This runs counter to the common wisdom based on the pioneering work of Belavin, Polyakov, Schwartz, and Tyupkin (BPST)[1] and supported by the outstanding work of Atiyah, Ward, Drinfeld, Hitchin, and Manin.[2] Nevertheless, Crewter[3] pointed out that solutions of the SDE with fractional topological charge might exist.

Here we argue that our solution does not actually contradict any of the existing theorems. BPST pointed out that the requirement of finite action implies that asymptotically in R^4 the gauge field A_μ should tend to a pure gauge $\partial_\mu g g^{-1}$, and with the assumption that g represents a continuous map of S^3 to SU(2) they concluded, using homotopy theory, that the topological charge must be an integer. However, finite action does not imply that this mapping must be continuous, and without continuity the concept of homotopy breaks down. If, however, g is not a continuous $S^3 \to S^3$ mapping, this automatically rules out the possibility of moving from R^4 to S^4 in the sense of the fiber-bundle approach, thus the theorems of Atiyah *et al.* do not apply here.[4] On the other hand, Uhlenbeck[5] has recently shown that from finite-action solutions of the Yang-Mills equations in R^4 pointlike singularities are removable, and so it is possible to extend this solution to S^4. Of course, this theorem is not applicable when the singularities of A_μ are not pointlike.

In fact, our solution has a singularity on a two-dimensional sphere (S_0^2) and it may be thought of as an extended object, to be contrasted with the pointlike structure of instantons. We may interpret it as a closed stringlike fluctuation of the vacuum, appearing at a certain instant (in Euclidean time) with zero radius, evolving to a maximal one, then shrinking back to zero radius again, and finally disappearing. Alternatively, since in Euclidean space there is no preferred time variable, we may describe our solution as a "balloon" (S_0^2) with a given radius appearing at a given instant and then disappearing again.

We now proceed to describe this solution in some detail. It is perhaps somewhat surprising that our solution is in the well-known Corrigan, Fairlie, 't Hooft, Wilczek (CFtHW)[6] *Ansatz*

$$A_\mu = \sigma_{\mu\nu}\,\partial^\nu \ln\rho, \tag{1}$$

where the SDE $F_{\mu\nu} = {}^*F_{\mu\nu}$ (Ref. 7) reduces to

$$\rho^{-1}\Box\rho = 0. \tag{2}$$

In this gauge we need two coordinate patches to describe A_μ.[7] In addition, even these two patches cover only $R^4 \setminus S_0^2$, and we have to define A_μ in the whole R^4 by an appropriate continuation, as will be explained later.

In the two patches the $A_\mu^{(i)}$'s are given by different superpotentials ρ_i:

$$A_\mu^{(i)} = \sigma_{\mu\nu}\,\partial^\nu \ln\rho_i, \quad i = 1, 2. \tag{3}$$

Now for our solution both ρ_i's depend only on z and r and they have the form

$$\rho_i = \rho_0 h_i, \tag{4}$$

which are given by

$$\rho_0 = (S^5 - S_-^5) r^{-1} S^{-5}, \quad h_i = S^5 (S^5 + S_-^5 + H_i)^{-1},$$
$$H_1 = [(S + S_-)^2 - 4\alpha^2]^{1/2} \{\tfrac{1}{4}[4\alpha^2 - (S - S_-)^2]^2 + S^2 S_-^2 - 12\alpha^2 (z-\beta)^2\}, \tag{5}$$
$$H_2 = [4\alpha^2 - (S - S_-)^2]^{1/2} \{\tfrac{1}{4}[4\alpha^2 - (S + S_-)^2]^2 + S^2 S_-^2 - 12\alpha^2 (z-\beta)^2\},$$

where

$$S=[(r+\alpha)^2+(z-\beta)^2]^{1/2}, \quad S_-=[(r-\alpha)^2+(z-\beta)^2]^{1/2}, \tag{6}$$

$\alpha>0$, and β is an arbitrary real number.

The two coordinate patches P_1 and P_2 are chosen in such a way that $A_\mu^{(i)}$ are free of any singularities in patch P_i; their projections on the (z, r) half plane are depicted in Fig. 1. The points A, B, C, and D on the z axis are excluded from the corresponding patches as the h_i functions have poles there; the line segments $(z=\beta,\ r<\alpha)$ and $(z=\beta,\ r>\alpha)$ are excluded from P_1 and P_2, respectively, as $\partial_\mu h_1$ and $\partial_\mu h_2$, respectively, are discontinuous there. Note that $S_0^{\,2}(z=\beta, r=\alpha)$ belongs to neither of the two patches.

In both domains of the overlapping region, the two $A_\mu^{(i)}$'s are connected by a continuous gauge transformation

$$A_\mu^{(1)}=\Omega A_\mu^{(2)}\Omega^{-1}+i\,\partial_\mu\Omega\,\Omega^{-1}, \tag{7}$$

where $\Omega=\exp[i\alpha(z,r)(\vec{\sigma}\cdot\vec{x}/2r)]$ with

$$\alpha(z,r)=\frac{\pi}{2}\operatorname{sgn}(z-\beta)+2\arctan\frac{R_1}{1-T_1}-2\arctan\frac{R_2}{1-T_2}$$

with R_i and T_i given by

$$R_1=[\operatorname{sgn}(z-\beta)/2S^5]H_2, \quad T_1=(2S^5)^{-1}H_1,$$
$$R_2=[\operatorname{sgn}(\beta-z)/2S^5]H_1, \quad T_2=(2S^5)^{-1}H_2. \tag{8}$$

We are forced to leave out $S_0^{\,2}$ from the overlapping region since the transition function Ω is not continuous there. However, from both patches the $A_\mu^{(i)}$'s can be continued back, with use of (3) and (4), to this sphere where $A_\mu^{(1)}=A_\mu^{(2)}$. Here we argue that the SDE are fulfilled even on $S_0^{\,2}$. If we extend the ρ_i's to the whole R^4 their derivatives become (singular) distributions; however, the main point here is to realize that on $S_0^{\,2}$ they give no contribution (in the sense that the appearing δ's are multiplied by coefficients vanishing on $S_0^{\,2}$). It is in this sense that our solution satisfies the SDE on the whole R^4. This situation is not unfamiliar because, in the case of the well-known instanton solutions, the SDE are satisfied in this gauge in a similar (distribution) sense, because of pointlike singularities in the connection (A_μ). Our case is different since $A_\mu^{(i)}$'s are free of singularities in $P_i\cup S_0^{\,2}$, but the transition function is not regular on $S_0^{\,2}$. It can be interpreted as a singularity of the bundle itself.

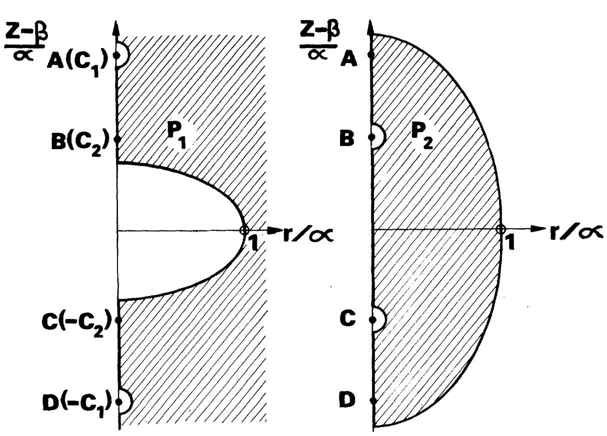

FIG. 1. The positions of the poles are given by $c_1=(1+2/\sqrt{5})^{1/2}$, $c_2=c_1^{\,-1}$.

We now proceed to calculate the topological charge

$$q=(8\pi^2)^{-1}\int d^4x\,F_{\mu\nu}{}^a\,{}^*F^{a\mu\nu}, \tag{9}$$

which is given by

$$q=-(16\pi^2)^{-1}\int d^4x\,\Box\Box\ln\rho. \tag{10}$$

The correct prescription for evaluating (10) is

$$q=-(16\pi^2)^{-1}\lim_{\epsilon\to 0}\int dz\,d\Omega\int_\epsilon^\infty r^2 dr\,\Box\Box\ln\rho, \tag{11}$$

since the action density $\frac{1}{4}F_{\mu\nu}{}^aF^{a\mu\nu}$ is regular at $r=0$. In fact, (10) should be interpreted as the sum of the integrals of $\Box\Box\ln\rho_i$ in $P_i\cup S_0^{\,2}$, subtracting the contribution coming from the overlapping region.

Now, observing that $\Box\Box\ln h_i$ is identically zero as a consequence of $(\partial_z^{\,2}+\partial_r^{\,2})\ln h_i=0$ in the domain of the integral, the topological charge is

$$q=-(16\pi^2)^{-1}\int dS^\mu\,\partial_\mu\Box\ln[(S^5-S_-^{\,5})/rS^5], \tag{12}$$

with use of the Gauss theorem. Equation (12) is readily evaluated and its value is found to be $\frac{3}{2}$, contributions to (12) coming from S^3 at $R\to\infty$ and the hypercylinder surrounding the z axis.

We now want to discuss the topological behavior of our solution. Since the gauge fixed by Eq. (1) is not suitable for discussing the asymptotics of the gauge fields at $R\to\infty$—$A_\mu^{(i)}$'s are vanishing faster than $\partial_\mu gg^{-1}$—we make a gauge transformation in the following way: First, in P_2 we carry out a gauge transformation S_2 on $A_\mu^{(2)}$, which makes $A_\mu^{(2)\prime}$ regular on the z axis. We now deform P_i to P_1' in such a way that P_1' does not contain the z axis. We then transform $A_\mu^{(1)}$ by S_1, where the S_i's are given by

$$S_i = \exp[-i\,\theta_i(z,r)(\vec{\sigma}\cdot\vec{x}/2r)], \tag{13}$$

with

$$\theta_i = \pi + 2\arctan\frac{R_i}{1-T_i} + 5\arctan\frac{\beta - z}{\alpha + r}.$$

As a result, the new transition function $\Omega^{(N)}$ is given as

$$\Omega^{(N)} = S_1 \Omega S_2^{-1} = \exp[i\tfrac{1}{2}\pi\,\mathrm{sgn}(z-\beta)(\vec{\sigma}\cdot\vec{x}/2r)].$$

The asymptotics of $A_\mu^{(i)\prime}$ in P_i' is

$$A_\mu^{(1)\prime} = i\,\partial_\mu g_1 g_1^{-1},$$

with $g_1 = \exp[i(3\varphi + \pi)(\vec{\sigma}\cdot\vec{x}/2r)]$, where $\varphi = \arctan z/r$;

$$A_\mu^{(2)\prime} = i\,\partial_\mu g_{2(+)} g_{2(+)}^{-1}$$

with $g_{2(+)} = \exp[i(3\varphi + \frac{1}{2}\pi)(\vec{\sigma}\cdot\vec{x}/2r)]$ for $z - \beta > 0$, while for $z - \beta < 0$,

$$A_\mu^{(2)\prime} = i\,\partial_\mu g_{2(-)} g_{2(-)}^{-1}$$

with $g_{2(-)} = \exp[i(3\varphi + \frac{3}{2}\pi)(\vec{\sigma}\cdot\vec{x}/2r)]$. Note that in P_2 the asymptotic domain consists of two disconnected parts, therefore, it is not surprising that $A_\mu^{(2)\prime}$ behaves differently in these regions. We remark that in this gauge on S_0^2 there is the same bundle type singularity as in the previous one.

One can now see the reason, in this gauge, for the fractional Pontryagin number: Although A_μ falls off as a pure gauge at infinity, it cannot be represented by a global pure gauge.[8]

The calculation of the topological charge (9) requires some care. Usually (9) is given by the surface integral of the topological current,

$$J_\mu = \mathrm{Tr}\,\epsilon_{\mu\nu\rho\sigma}(A^\nu \partial^\rho A^\sigma + i\tfrac{2}{3} A^\nu A^\rho A^\sigma),$$

on S^3 at infinity. Since there are several patches in our case, when one applies the Gauss theorem there are additional contributions coming from the boundaries. However, shrinking the overlapping region to the hypersurface $\alpha^2 - r^2 + (z - \beta)^2 = 0$, which means that on the asymptotic S^3, P_{2+}', P_1', and P_{2-}' are defined as $\frac{1}{2}\pi \geq \varphi \geq \frac{1}{4}\pi$, $\frac{1}{4}\pi \geq \varphi \geq -\frac{1}{4}\pi$, and $-\frac{1}{4}\pi \geq \varphi \geq -\frac{1}{2}\pi$, respectively, the total contribution to the Pontryagin number comes from infinity only.

Higher derivatives of gauge-invariant quantities (e.g., the action density) will be singular on S_0^1, which means that this surface singularity cannot be moved by gauge transformations. Therefore, our solution depends on eight parameters: four corresponding to the location of the center, one to the radius of S_0^1, and three to the direction of the symmetry axis in R^4.

The existence of this solution may be relevant for an understanding of the structure of the quantum-chromodynamic vacuum,[9] and it may provide a solution of the U(1) problem as advocated by Crewther.[3] Further clarification is needed of the relevance of these closed stringlike fluctuations to the confinement problem.

Since our solution is cylindrically symmetric, it can be related to Witten's *Ansatz*,[10] which corresponds to an Abelian Higgs model defined on a two-dimensional pseudosphere. As it was pointed out,[11] on a two-dimensional noncompact manifold in a U(1) gauge theory there are configurations with noninteger topological charge even in the presence of fermionic matter.

We would like to mention that solutions of the SDE with topological charge other than half-integer exist. Work is in progress in this direction and we shall present these results later.

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[7] Our notations and conventions are as follows:

$$A_\mu = A_\mu^a \tfrac{1}{2}\sigma^a, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu];$$

$$\sigma_{ij} = (1/4i)[\sigma_i, \sigma_j], \; j \neq 0; \; \sigma_{i0} = \tfrac{1}{2}\sigma_i;$$

$$x^0 = z, \; r = (x^2 + y^2 + t^2)^{1/2}, \; R = (z^2 + r^2)^{1/2}.$$

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