Exact Results for the Two-Dimensional One-Component Plasma

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At some special temperature T_0, the distribution functions of a two-dimensional one-component plasma are explicitly computed up to the four-body one. The correlations have a Gaussian falloff. The distribution functions at T_0 are used for building a temperature expansion around T_0.

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A one-component plasma is a system of N identical particles of charge e embedded in a uniform neutralizing background of opposite charge. In two dimensions, the Coulomb interaction potential between two particles at a distance r from one another is

$$e^2 v(r) = -e^2 \ln(r/L), \tag{1}$$

where L is a length scale. If one assumes the particles to be confined in a disk of radius R, the total potential is[1]

$$V = -\frac{3N^2e^2}{8} + \frac{Ne^2}{2}\ln\frac{R}{L} + \frac{Ne^2}{2}\sum_i \left(\frac{r_i}{R}\right)^2 - e^2 \sum_{i>j} \ln\frac{r_{ij}}{R}, \tag{2}$$

where $\vec{r}_i$ is the position of particle i (the origin is chosen at the center of the disk). By using the scaled variables $\vec{z}_i = N^{1/2}\vec{r}_i/R$, one easily shows that the excess free energy per particle, which must have a well-behaved thermodynamic limit,[2] is necessarily of the form

$$F_{\text{exc}}/N = -\tfrac{1}{4}e^2 \ln(\pi\rho L^2) + f(T), \tag{3}$$

where $\rho = N/\pi R^2$ is the number density, and $f(T)$ is some function of the temperature alone. Therefore, the equation of state has the simple form[3]

$$p = (k_B T - \tfrac{1}{4}e^2)\rho, \tag{4}$$

where k_B is Boltzmann's constant and T is the temperature. More information can be obtained at the special temperature $T_0 = e^2/2k_B$; recently, the free energy has been exactly computed[4] at T_0, with the result

$$F_{\text{exc}}/N = -\tfrac{1}{4}e^2 \ln(\pi\rho L^2) + e^2[\tfrac{1}{2} - \tfrac{1}{4}\ln(2\pi)]. \tag{5}$$

In the present Letter, the distribution functions at T_0 are explicitly computed up to the four-body one and used for building a temperature expansion around T_0.

Using for $\vec{z}_i$ polar coordinates (z_i, θ_i), one obtains at T_0 from (2) a Boltzmann factor

$$\exp(-V/k_B T_0) = A \exp(-\sum_i z_i^2)\left|\prod_{i>j}(Z_i - Z_j)\right|^2, \tag{6}$$

where A is a constant and $Z_i = z_i \exp(i\theta_i)$. This expression (6), which also occurs in the theory of random matrices, can be integrated upon variables $\vec{z}_i (0 \leq z_i \leq \sqrt{N})$ by expanding the Vandermonde determinant $\prod(Z_i - Z_j)$. One obtains the partition function[5,6]

$$\int \exp(-V/k_B T_0) d^2z_1 \cdots d^2z_N = A\pi^N N! \prod_{j=1}^{N} \gamma(j,N), \tag{7}$$

where

$$\gamma(j,N) = \int_0^{\sqrt{N}} \exp(-z^2) z^{2(j-1)} 2z\, dz \tag{8}$$

$$= \int_0^N e^{-t} t^{j-1} dt \tag{9}$$

is the incomplete gamma function; in Ref. 4, (8) was used for computing the free energy (5). One can also obtain the n-body distribution functions[5,6]

$$g(1, \ldots, n) = \exp(-\sum_{i=1}^{n} z_i^2) \operatorname{Det}[K_N(Z_i Z_j^*)]_{i,j=1,\ldots,n}, \tag{10}$$

where

$$K_N(Z_i Z_j^*) = \sum_{l=1}^{N} \frac{(Z_i Z_j^*)^{l-1}}{\gamma(l,N)}. \tag{11}$$

In the thermodynamic limit $N \to \infty$, $\gamma(l,N) \to (l-1)!$, and $K_N(Z_i Z_j^*) \to \exp(Z_i Z_j^*)$ [the terms with l close to N make no trouble since $\gamma(N,N) \sim \frac{1}{2}(N-1)!$]. In this limit, one obtains from (10) the following explicit distribution functions: The one-body density $\rho(1) = \rho g(1)$ has the constant value ρ. The pair distribution function is

$$g(1,2) = 1 - \exp(-\pi\rho r_{12}^2), \tag{12}$$

where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$. The three-body distribution function is

$$g(1,2,3) = 1 - \exp(-\pi\rho r_{12}^2) - \exp(-\pi\rho r_{23}^2) - \exp(-\pi\rho r_{31}^2) + 2\exp[-\tfrac{1}{2}\pi\rho(r_{12}^2 + r_{23}^2 + r_{31}^2)]\cos[2\pi\rho A(1,2,3)], \tag{13}$$

where $A(1,2,3)$ is the area of the triangle formed by particles $(1,2,3)$. The four-body distribution function is

$$\begin{aligned} g(1,2,3,4) = 1 &- \exp(-\pi\rho r_{12}^2) - \cdots \\ &+ \exp[-\pi\rho(r_{12}^2 + r_{34}^2)] + \cdots \\ &+ 2\exp[-\tfrac{1}{2}\pi\rho(r_{12}^2 + r_{23}^2 + r_{31}^2)]\cos[2\pi\rho A(1,2,3)] + \cdots \\ &- 2\exp[-\tfrac{1}{2}\pi\rho(r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{41}^2)]\cos[2\pi\rho A(1,2,3,4)] - \cdots, \end{aligned} \tag{14}$$

where

$$A(1,2,3,4) = \tfrac{1}{2}|\vec{r}_{13} \times \vec{r}_{24}| \tag{15}$$

is the area of the quadrilateral formed by particles $(1,2,3,4)$ (or a difference of areas if that quadrilateral has intersecting sides).

These distribution functions are translationally invariant. They show neither long-range order nor quasi-long-range[7] order: At T_0, the system is a fluid. A somewhat surprising result is that *the correlations have a Gaussian falloff* rather than the exponential one which is found[8,9] in the high-temperature Debye approximation. One easily checks that the distribution functions obey the perfect screening and other sum rules.[10]

It is convenient to express the temperature T through the dimensionless coupling constant $\Gamma = e^2/k_BT$; at T_0, $\Gamma = 2$. Using the distribution functions at $\Gamma = 2$, one can now build expansions in powers of $\Gamma - 2$. The pair distribution function at Γ is

$$\begin{aligned} g(1,2;\Gamma) = g(1,2) + (\Gamma - 2)\{&-g(1,2)v(1,2) - 2\rho\int[g(1,2,3) - g(1,2)]v(1,3)d3 \\ &- \tfrac{1}{2}\rho^2\int[g(1,2,3,4) - g(1,2)g(3,4) - g(1,2,3) - g(1,2,4) + 2g(1,2)] \\ &\times v(3,4)d3\,d4\} + \cdots, \end{aligned} \tag{16}$$

where the absence of Γ in the arguments of g means $\Gamma = 2$. Equation (16) differs from the usual perturbation expansion for neutral fluids[11] in two ways. First, there are additional terms due to the particle-background interaction. Second, as a consequence of perfect screening, there is *no* term of order $1/N$ in $g(1,2,3,4) - g(1,2)g(3,4)$ when the pairs $(1,2)$ and $(3,4)$ are widely separated [for neutral fluids, such terms give additional finite contributions to (16) in the thermodynamic limit]. One finds from (16)

$$\begin{aligned} g(r;\Gamma) = 1 - \exp(-\pi\rho r^2) + (\Gamma - 2)\{&-\exp(-\pi\rho r^2)[\ln(\pi\rho r^2) + c] \\ &+ \mathrm{Ei}(-\pi\rho r^2) - \tfrac{1}{2}\mathrm{Ei}(-\tfrac{1}{2}\pi\rho r^2) + \tfrac{1}{2}\exp(-\pi\rho r^2)\,\mathrm{Ei}(\tfrac{1}{2}\pi\rho r^2)\} + \cdots, \end{aligned} \tag{17}$$

where $C = 0.5772\ldots$ is Euler's constant and Ei is the experimental-integral function. It can be checked that $g(r;\Gamma)$ obeys the usual sum rules[10] to order $\Gamma - 2$.

At small r, for any value of Γ, $g(r;\Gamma)$ should exhibit a bare-potential factor $\exp(\Gamma\ln r)$; this factor gives a $\ln r$ behavior both in the Debye approximation[8] valid near $\Gamma = 0$ and in the present expansion (17) around $\Gamma = 2$. At large r, the bracket in (17) behaves like $2\exp(-\frac{1}{2}\pi\rho r^2)/\pi\rho r^2$; thus, the correction of order $\Gamma - 2$ introduces a correlation which is still Gaussian-like at long distance, but with a longer range. Further corrections of increasing order in $\Gamma - 2$ would have increasing ranges; presumably, if the whole series could be summed, one would recover near $\Gamma = 0$ the exponential falloff of the Debye approximation. Conversely, if the full Debye expansion around $\Gamma = 0$ could be summed, it should give a Gaussian at $\Gamma = 2$.

For large values of Γ, one expects oscillations in $g(r)$. Since the correction of order $\Gamma - 2$ in (17) has the sign of $\Gamma - 2$, it is tempting to conjecture that $g(r)$ changes from a monotonic to an oscillating behavior precisely at $\Gamma = 2$.

Knowing $g(r;\Gamma)$ up to the order $\Gamma - 2$, one can compute the free energy up to the order $(\Gamma - 2)^2$, or, equivalently, the internal energy U and the specific heat c at $\Gamma=2$. Their excess parts per particle are found to be

$$U_{\text{exc}}/N=-\tfrac{1}{4}e^2\ln(\pi\rho L^2)-\tfrac{1}{4}e^2C, \qquad (18)$$

and

$$c_{\text{exc}}/N=k_{\text{B}}(\ln 2-\pi^2/24). \qquad (19)$$

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[1] R. R. Sari, D. Merlini, and R. Calinon, J. Phys. A: Gen. Phys. 9, 1539 (1976).

[2] R. R. Sari and D. Merlini, J. Stat. Phys. 14, 91 (1976).

[3] E. H. Hauge and P. C. Hemmer, Phys. Norv. 5, 109 (1971), and references quoted therein.

[4] A. Alastuey and B. Jancovici, to be published.

[5] J. Ginibre, J. Math. Phys. 6, 440 (1965).

[6] C. L. Mehta, *Random Matrices* (Academic, New York, 1967).

[7] D. R. Nelson, Phys. Rev. B 18, 2318 (1978), and references quoted therein.

[8] C. Deutsch and M. Lavaud, Phys. Rev. Lett. 31, 921 (1973), and Phys. Rev. A 9, 2598 (1974).

[9] C. Deutsch, H. E. DeWitt, and Y. Furutani, Phys. Rev. A 20, 2631 (1979).

[10] See, e.g., M. Baus and J. P. Hansen, Phys. Rep. 59, 1 (1980).

[11] See, e.g., D. Henderson and J. A. Barker, in *Liquid State Chemistry*, edited by Douglas Henderson, Volume 8A of Physical Chemistry (Academic, New York, 1971).

Quantum-Statistical Metastability

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Consider a system rendered unstable by both quantum tunneling and thermodynamic fluctuation. The tunneling rate Γ, at temperature β^{-1}, is related to the free energy F by $\Gamma=(2/\hbar)\,\text{Im}F$. However, the classical escape rate is $\Gamma=(\omega\beta/\pi)\,\text{Im}F$, $-\omega^2$ being the negative eigenvalue at the saddle point. A general theory of metastability is constructed in which these formulas are true for temperatures, respectively, below and above $\omega\hbar/2\pi$ with a narrow transition region of $O(\hbar^{3/2})$.

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Consider a system with a localized metastable ground state and a saddle point through which the system can escape to the true ground state. A simple example is a particle in the one-dimensional (1D) potential of Fig. 1. I shall at first concentrate on this example and then generalize to an arbitrary system (which may be a field theory). One may safely assume that both the ground-state energy, $\frac{1}{2}\hbar\omega_0\;[V''(x_0)=\omega_0^2]$, and the temperature are small compared to the barrier height, V_0; otherwise, the system would not be metastable.

At temperatures small compared to $\hbar\omega_0$ the particle is mainly in the low-lying metastable states. These have wave functions that vanish at $-\infty$, are standing waves normalized to 1 in the well, and give an exponentially small probability current, J, at positive x, which is identified with the decay rate, $\Gamma(E)$. The nonconservation of J requires that E have an (exponentially small) imaginary part[1] which obeys $\Gamma=(2/\hbar)\,\text{Im}E$. Taking a Boltzmann average of $\Gamma(E)$, we find $\Gamma=(2/\hbar)\,\text{Im}F$, to lowest order in exponentially small quantities.

At temperatures large compared to $\hbar\omega_0$ (but still small compared to V_0) we would expect classical thermodynamic fluctuations to dominate. The classical rate[2] is calculated by setting up a

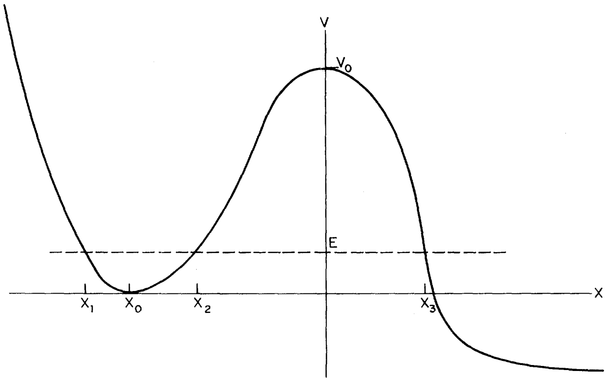

FIG. 1. The potential for a 1D metastable system.