variables and commute with all else. Then

$$H = -\tfrac{1}{2}\sum (a\gamma_2 + b\mu_2 + c\lambda_2) \tag{27}$$

has self-triality. One then satisfies the anticommutation algebra either with $\mu_2 = \gamma_3$ and $\lambda_2 = \gamma_{34}\gamma_{12}'$ and deals with $H(\gamma)$ or one sets $\gamma_2 = 2i\psi_1\psi_2$, $\mu_2 = 2i\psi_1\psi_3$, and $\lambda_2 = 2i\psi_2\psi_3'$ and gets the quadratic form $H(\psi)$.[10] But note that in all these cases the existence of a fermion solution to the algebra is of no use unless we also have a solution in terms of σ or γ matrices, for the free Fermi theory is interesting only because it maps onto a spin problem.

Since completing this analysis, I have become aware of the work of Srednicki, Fradkin, and Suskind,[11] which also arrives at some of the notions I discussed towards the end of this work on the role of fermion variables in spin systems.

A longer paper discussing many more issues is in preparation.

It is a pleasure to acknowledge useful dialogues with F. Gürsey, A. Chodos, R. Pisarski, E. Witten, and especially I. Bars and E. Fradkin. This research was supported in part by the U. S. Department of Energy under Contract No. EY-76-C-02-3075.

Note added.—I have been informed by Professor E. Lieb of related work by Bashilov and Pokrovsky.[12]

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[10]In completing the μ and λ sets, we must also attain the following symmetry in the transformation functions: $\gamma(\lambda)$ must be the same function as $\mu(\lambda)$ if H is to be invariant under triality. There exist other sets of variables isomorphic to γ that do not satisfy this.

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Black Holes Do Evaporate Thermally

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(Received 7 November 1980)

A careful consideration of the propagation of null geodesics in a black-hole geometry modified by Hawking-radiation back reaction shows that the event horizon is stable and shifted only very slightly in radius from what is expected in a vacuum background, contrary to a recent claim in the literature.

PACS numbers: 04.20.Cv, 95.30.Sf, 97.60.Lf

The original derivation of the Hawking radiation from black holes[1,2] assumed that the black hole is static or stationary for the purpose of calculating the rate and properties of the radiation. Overall energy conservation requires that the quantum fluctuations in the vicinity of the event horizon responsible for the radiation produce an energy-momentum tensor whose expectation value corresponds to a negative energy flux into the black hole. Several attempts[2,3-6] have been made at a direct calculation of the effective energy-momentum tensor, which in the absence of a manageable quantum theory of gravity can be used in a semiclassical approximation to find the evolution of the black hole. The geometry, through the classical Einstein equations, and therefore the propagation of null geodesics is modified from that expected in a vacuum black-hole metric.

Tipler[7] recently considered this back reaction for spherically symmetric black holes and concluded that the event horizon is unstable on a time scale which even for a 1-solar-mass black hole is as short as 1 sec. There is, in fact, no such instability. I show in this paper that the usual quasistationary approximation is indeed justified as long as the black-hole mass M is much greater than the Planck mass $m_P \equiv (\hbar c/G)^{1/2} \sim 10^{-5}$ g. The null hypersurface which forms the event horizon is, as a result of the back reaction, shifted to a slightly smaller radius than the vacuum value $r = 2M$ $(G = c = 1)$, by a fractional amount the order of $(m_P/M)^2$. Such a small change is the order of the quantum fluctuations in the geometry[2] and is physically negligible.

Like Tipler, I assume the classical background geometry is spherically symmetric and choose units such that $G = c = 1$. Convenient coordinates are the circumferential radius r (so the area of a two-sphere is $4\pi r^2$) and an advanced time v which is constant on ingoing radial null geodesics. These reduce to Eddington-Finkelstein coordinates[8] in vacuum and are regular on the future event horizon. A physically general line element has the form

$$ds^2 = -(1 - 2m/r)e^{2\psi}dv^2 + 2e^{\psi}dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{1}$$

The quantity m has a coordinate-invariant meaning, in that $1 - 2m/r$ is the square of the gradient of the circumferential radius. Its value as $r \to \infty$ is the total gravitational mass of the system.

The Einstein field equations relate $m(v, r)$ and $\psi(v, r)$ to the energy-momentum tensor $T^{\mu\nu}$ in a remarkably simple way:

$$\partial m/\partial r = 4\pi r^2 T^v{}_v, \tag{2}$$

$$\partial m/\partial v = 4\pi r^2 T^r{}_v, \tag{3}$$

$$\partial\psi/\partial r = 4\pi r\, T_{rr} = 4\pi r\, e^{\psi} T^v{}_r. \tag{4}$$

To date there is still some ambiguity in direct calculations of the expectation value of the regularized quantum energy-momentum tensor in a Schwarzschild background.[6] Fortunately, energy conservation relates the Hawking energy flux at $r \gg 2m$ to the energy flux at $r \sim 2m$, and to first order in the back reaction the null geodesic trajectories near $r = 2m$ depend only on the energy flux there. Consistent with the direct calculations, I assume that in the vacuum appropriate to a black hole formed by gravitational collapse[9] an observer falling across $r \sim 2m$ measures a regular energy-momentum tensor.

Consider an observer at circumferential radius r whose rate of change of radius with proper time is $dr/d\tau$. Denote the components of the energy-momentum tensor in the observer's local reference frame by $T^{(\alpha)(\beta)}$. The relation of the local frame to the (v, r) coordinates is defined mathematically by the tetrad of orthonormal reactors $e_{(\alpha)}{}^{\mu}$, with

$$e_{(0)}{}^{\mu} = [e^{-\psi}(\Gamma - dr/d\tau)^{-1}, dr/d\tau, 0, 0], \tag{5a}$$

$$e_{(1)}{}^{\mu} = [e^{-\psi}(\Gamma - dr/d\tau)^{-1}, \Gamma, 0, 0], \tag{5b}$$

where

$$\Gamma \equiv [1 - 2m/r + (dr/d\tau)^2]^{1/2}. \tag{6}$$

Since the tetrad vector components are regular at $r \sim 2m$ if $dr/d\tau \sim -1$, as by assumption are the $T^{(\alpha)(\beta)}$, the coordinate components

$$T^{\mu\nu} = e_{(\alpha)}{}^{\mu} e_{(\beta)}{}^{\nu} T^{(\alpha)(\beta)}$$

are regular at $r \sim 2m$ and comparable in magnitude for $\mu, \nu = 0, 1$.

The energy-conservation equation, slightly simplified with the help of Eqs. (3) and (4), is

$$T_v{}^{\mu}{}_{;\mu} = r^{-2}[r^2 T^r{}_v]_{,r} + [T^v{}_v]_{,v} = 0. \tag{7}$$

The energy-momentum tensor is expected to be quasistationary, so the time derivative in Eq. (7) can be neglected, out to radii small compared with both the evaporation time of the black hole and the time since the black hole was formed. An observer at constant $r \gg 2m$, but well within the quasistationary region, measures an energy flux

$$T^{(0)(1)} = L_H/4\pi r^2 \simeq -e^{-\psi} T^r{}_v. \tag{8}$$

By assuming $T^v{}_r \sim T^r{}_v$, Eqs. (4) and (7) give

$$\partial\psi/\partial r \sim L_H/r \tag{9}$$

in the quasistationary region.

It makes sense to *define* the black-hole mass at a given time v to be

$$M(v) \equiv m(v, r = 2m), \tag{10}$$

and to scale v so $\psi = 0$ at $r = 2m$. In a Schwarzschild background the Hawking luminosity L_H scales as[1,2]

$$L_H = B(m_P/M)^2. \tag{11}$$

The coefficient B depends on the number of spin states of radiatable particles and barrier penetration factors. As long as the temperature of the black hole is less than 200 MeV or so, M

$>10^{14}$ g, calculations by Page[10] show that B is nearly constant, $B\sim 10^{-3}$. I adopt Eq. (11) to calculate the first-order effects of back reaction. Since ψ and fractional variations of m on radial and time scales the order of M are themselves first order in L_H, integration of Eq. (7) to the upper limit $r=2M$ from a lower limit r, started with Eq. (8), not too large compared with M, if the slow time dependence of M in applying Eq. (11) is ignored, gives at $r\simeq 2M$

$$dM/dv \simeq \partial m/\partial v = 4\pi r^2 T^r{}_v \simeq -L_H = -B(m_P/M)^2. \qquad (12)$$

This integrates to

$$M \simeq [3B\, m_P^2(v_0 - v)]^{1/3} \qquad (13)$$

if, for simplicity, B is assumed constant. The integration constant v_0 is the advanced time at which the black hole disappears.

As raised by Tipler,[7] the question of the stability or more precisely the *existence* of the event horizon is the question of whether a null geodesic can remain at $r\sim 2m$ during the whole evaporation process to mark the inner boundary of what is visible from infinity.[11] The *trajectories* of the outward radial null geodesics are best determined directly from the condition that the tangent vector be null,

$$e^{-\psi} dr/dv = \tfrac{1}{2}(1-2m/r). \qquad (14)$$

The right-hand side of Eq. (14) is approximated *uniformly* through first order in L_H for $2m \lesssim r < \infty$ by setting $m(v,r)\simeq M(v)$. When $|r/2m-1| \ll 1$, m and M differ only in order $(L_H)^2$, by the definition of M and the regularity of m, and so the *fractional* error in $1-2m/r$ is of order L_H. The potential ψ is of order L_H at $r/2M-1\sim 1$ and even as $r\to\infty$ is only of order $L_H \ln(L_H)$.

Now consider Eq. (14) at $|r/2m-1|\ll 1$, in order to calculate the initial deviation of the geodesics from the event horizon. Let

$$z \equiv r/2M - 1. \qquad (15)$$

To first order in L_H and z, and with the help of Eq. (12), Eq. (14) can be written

$$dz/dv = z/4M + Bm_P^2/M^3. \qquad (16)$$

With the substitution $z\simeq 2dr/dv$ from Eq. (14), Eq. (16) would become a second order equation for r identical to Eq. (7) of Tipler.[7] Of course, not all solutions of the second-order equation are compatible with Eq. (14).

Since Eq. (16) is linear in z it has the straightforward solution

$$z = \exp\left(\int_{v_1}^{v} \frac{dv'}{4M}\right)\left[z_1 + m_P^2\int_{v_1}^{v} \frac{B\,dv'}{M^3}\exp\left(-\int_{v_1}^{v'}\frac{dv''}{4M}\right)\right], \qquad (17)$$

where z_1 is the initial value of z, at $v=v_1$. As v increases from v_1 the second term inside the square brackets rapidly approaches a constant value. For almost all values of z_1 the exponential factor in front causes z to increase rapidly in magnitude, e-folding on a dynamic time scale. Even if $|z_1|\sim L_H$ to begin with, in an astronomically short time $|z|$ will become of order unity and the null geodesic will begin its escape to $r \gg 2m$ or, if $z<0$, its fall to the presumed singularity at $r=0$. It is this behavior which led Tipler to claim the event horizon is unstable.

The exponential runaway has nothing to do with back reaction. It is present even if $\hbar = m_P^2 = 0$. The only difference with back reaction is that the critical value of z_1 which divides inward-diverging from outward-diverging geodesics is negative instead of zero. The event horizon is *defined* as the boundary between trapped and escaping null geodesics.[11] The event horizon consists of those null geodesics which remain at $|z|\ll 1$ at least until M has decreased to m_P. Tipler's claim that no such solutions exist is false.

The values of z_1 which generate the event horizon are

$$z_1 = -m_P^2\int_{v_1}^{v_2}(B\,dv'/M^3)\exp\left[-\int_{v_1}^{v'}(dv''/4M)\right], \qquad (18)$$

where v_2 is a time at which $M\sim m_P$. The precise value of v_2 is unimportant. When Eq. (18) is substituted back into Eq. (17), the result is

$$z_h(v) \equiv (r_h/2M-1) = -m_P^2\int_{v}^{v_2}(B\,dv'/M^3)\exp\left[-\int_{v}^{v'}(dv''/4M)\right]. \qquad (19)$$

The uncertainty in v_2 associated with the breakdown of the classical geometry at $M\sim m_P$ produces an uncertainty in $z_h(v)$ which is exponentially *small* until v is within $\Delta v\sim m_P$ of v_2. In this physically appropriate sense the event horizon is very *stable.*

An approximation to the solution (19), based on the fact M is nearly constant on dynamical time scales for $M \gg m_{\rm P}$, is

$$z_h \simeq -B\frac{m_{\rm P}^2}{M^3}\int_v^{v_2} dv' \exp[-(v'-v)/4M] \simeq -4L_{\rm H}, \tag{20}$$

since $M \gg m_{\rm P}$ also implies $(v_2 - v)/4M \gg 1$. A direct route to Eq. (20) is to neglect dz/dv in Eq. (16). Note that the change in z_h due to back reaction is no larger than the fractional change in m between the horizon and a radius where the Hawking flux is well defined.

It is important to remember that Eq. (20) is just a (very good) approximation. Since z_h decreases with decreasing M and increasing v, the true z_h must be very slightly (of order $L_{\rm H}^2$) less than the z_h given by Eq. (20) to give the required slightly negative value of dz/dv in Eq. (16).

What of the tendency of negative energy density to bend null geodesics outward? With an *affine*, rather than an advanced-time parametrization of the geodesic, the geodesic deviation equation corresponding to Eq. (7) of Tipler is

$$d^2r/d\lambda^2 + 4\pi r T_{\mu\nu}(dx^\mu/d\lambda)dx^\nu/d\lambda = 0. \tag{21}$$

When $|r/2m - 1| \ll 1$, by Eq. (14) $|dr/d\lambda| \ll dv/d\lambda$ and

$$T_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \simeq T_{vv}\left(\frac{dv}{d\lambda}\right)^2 \simeq T^r{}_v e^{\psi}\left(\frac{dv}{d\lambda}\right)^2 \simeq -\frac{e^{2\psi}L_{\rm H}}{4\pi r^2}\left(\frac{dv}{d\lambda}\right)^2. \tag{22}$$

Indeed $d^2r/d\lambda^2 > 0$, but the slight outward curvature as measured by the affine parameter is unable to overcome the initial negative value of $dr/d\lambda$ when $z < z_h$. An exponential relation between affine parameter and advanced time is why the linear homogeneous solution of Eq. (21) is consistent with the exponential homogeneous solution of Eq. (16). With the help of Eq. (14), Eq. (21) becomes an equation for $d\lambda/dv$ which in the limit $|r/2m - 1| \ll 1$ has the solution

$$d\lambda/dv \simeq C\exp[\textstyle\int^v (dv'/4M)]. \tag{23}$$

Now that the quasistationary nature of the horizon in the presence of back reaction has been confirmed, it is straightforward to check that the effective surface gravity of the black hole, which translates into the temperature of the Hawking radiation as measured by an observer at radius r, $2M \ll r \ll M/L_{\rm H}$, is unchanged by back reaction if the mass in the static background is identified with the current mass of the evaporating black hole.

In conclusion, a semiclassical treatment of back reaction from the Hawking radiation entirely supports the validity of calculations of the radiation based on a static background geometry. By the time the back-reaction corrections become appreciable, at $M \sim m_{\rm P}$, the classical concept of an event horizon is no longer meaningful.

A less general refutation of Tipler, based on a specific model for the effective energy-momentum tensor near the horizon, has been published by Hajicek and Israel.[12]

I am grateful to James B. Hartle and David G. Boulware for helpful discussions.

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