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Solvable Models with Self-Triality in Statistical Mechanics and Field Theory

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The notion of self-duality is extended to self-triality. One example from spin systems is given and *completely solved* by use of fermion variables. It is then shown that the O(8) Gross-Neveu model has self-triality: The Lagrangian $\mathcal{L}(\psi)=\mathcal{L}(R)=\mathcal{L}(L)$, where ψ is the original fermion while R and L are two types of kinks that occur dynamically. The anatomy of self-duality (triality) in the Ising and present examples is exposed as is the origin of the fermionic solutions.

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Let us begin by recalling a model with self-duality, the quantum Ising model in one dimension[1,2]:

$$H=-\tfrac{1}{2}a\sum_n \sigma_3\sigma_3{}'-\tfrac{1}{2}b\sum_n\sigma_1, \tag{1}$$

where a and b are parameters, σ_i and $\sigma_j{}'$ are Pauli matrices at sites n and $n+1$, respectively. If one introduces the dual variables[1-3]

$$\mu_3(n)=\prod_{m=-\infty}^{n}\sigma_1(m)\equiv(\overleftarrow{\prod}\sigma_1)\sigma_1(n), \tag{2}$$

$$\mu_1(n)=\sigma_3\sigma_3{}', \tag{3}$$

one finds

$$H=-\tfrac{1}{2}a\sum_n\mu_1-\tfrac{1}{2}b\sum_n\bar{\mu}_3\mu_3, \tag{4}$$

where $\bar{\mu}_3(n)=\mu_3(n-1)$. Since μ and σ are isomorphic, H has *self-duality* and $H(a,b)=H(b,a)$. Not only does the exchange $a\leftrightarrow b$ or $\lambda=b/a\rightarrow 1/\lambda$ facilitate computations, it has an interesting interpretation as the change from order to disorder or kink variables.[1-3]

Here I illustrate self-triality through two examples. One is a spin problem where $H(a,b,c)$ goes into $H(b,c,a)$ or $H(c,a,b)$ upon changing to either of *two* dual variables. The other is the O(8) Gross-Neveu model in which the original fermion ψ generates two types of kinks R and L. I show that $\mathcal{L}(\psi)=\mathcal{L}(R)=\mathcal{L}(L)$, $\mathcal{L}$ being the Lagrangian. Since $\mathcal{L}$ is *invariant*, we are at the self-triality point ($a=b=c$). The spin problem is completely solved and seems related to Baxter's three-color problem.[4] The O(8) model has already been solved at the S-matrix level.[5]

Finally I show how to construct a class of models with self-duality (triality) and how from the very construction we see the existence of a fermionic solution and the relevant fermionic variables.

The spin system.—The Hamiltonian is

$$H=-\tfrac{1}{2}a\sum_n\gamma_2-\tfrac{1}{2}b\sum_n\gamma_3-\tfrac{1}{2}c\sum_n\gamma_{34}\gamma_{12}{}', \tag{5}$$

where γ_i are Hermitian O(4) Dirac matrices obeying $\{\gamma_i,\gamma_j\}=2\delta_{ij}$ and $\gamma_{ij}=\gamma_i\gamma_j$ are anti-Hermitian matrices with $\gamma_{ij}{}^2=-1$. [We may also view them all as SU(4)≅O(6) generators.] Changing to either

$$\mu_2=\gamma_3,\quad \mu_3=\gamma_{34}\gamma_{12}{}',$$
$$\mu_{12}=(\overleftarrow{\prod}i\gamma_{14})i\gamma_2\equiv\prod_{m=-\infty}^{n-1}[i\gamma_{14}(m)]i\gamma_2(n), \tag{6}$$
$$\mu_{34}=(\overleftarrow{\prod}\gamma_{14})\gamma_{14},$$

or

$$\lambda_2=\bar{\gamma}_{34}\gamma_{12}, \quad \lambda_3=\gamma_2,$$
$$\lambda_{12}=\gamma_{14}\vec{\prod}(i\gamma_{14})\equiv\gamma_{14}(n)\prod_{m=n+1}^{\infty}[i\gamma_{14}(m)], \quad (7)$$
$$\lambda_{34}=i\gamma_3\vec{\prod}(i\gamma_{14}),$$

which are isomorphic to the γ's we obtain

$$H(a,b,c;\gamma)=H(b,c,a;\mu)=H(c,a,b;\lambda). \quad (8)$$

We may, of course, set $\gamma=\mu=\lambda$ in the above since all variables are isomorphic.

To solve the model, I define three self-adjoint fermion variables

$$\psi_1=(1/\sqrt{2})\gamma_4\vec{\prod}(i\gamma_{14}),$$
$$\psi_2=(1/\sqrt{2})i\gamma_{34}\vec{\prod}(i\gamma_{14}), \quad (9)$$
$$\psi_3=(1/\sqrt{2})i\gamma_{24}\vec{\prod}(i\gamma_{14}),$$

which obey the standard Majorana algebra

$$\{\psi_i(m),\psi_j(n)\}=\delta_{ij}\delta_{mn}. \quad (10)$$

In terms of these

$$H=-a\sum_n i\psi_1\psi_3-b\sum_n i\psi_1\psi_2+c\sum_n i\psi_2\psi_3'. \quad (11)$$

To diagonalize the quadratic form, I Fourier expand

$$\psi_i(n)=(2N+1)^{-1/2}\sum_k[c_i(k)e^{-ikn}+c_i^\dagger(k)e^{ikn}], \quad (12)$$

where

$$k=2n\pi/(2N+1), \quad n=0,1,2,\ldots,N. \quad (13)$$

One may check that

$$\{c_i(k),c_j^\dagger(l)\}=\delta_{ij}\delta_{kl}. \quad (14)$$

In k space,

$$H=\sum_{k=0}^{\pi}(c_1^\dagger \quad c_2^\dagger \quad c_3^\dagger)\begin{pmatrix}0 & -ib & -ia\\ ib & 0 & ice^{-ik}\\ ia & -ice^{ik} & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}, \quad (15)$$

$$=\sum_{k=0}^{\pi}\Lambda_1(k)\eta_1^\dagger\eta_1+\Lambda_2(k)\eta_2^\dagger\eta_2+\Lambda_3(k)\eta_3^\dagger\eta_3, \quad (16)$$

where Λ_i are roots of

$$\Lambda(\Lambda^2-a^2-b^2-c^2)+2abc\sin k=0 \quad (17)$$

and are given by

$$\Lambda_1=[(a^2+b^2+c^2)/3]^{1/2}2\cos\theta,$$
$$\Lambda_{2,3}=-\tfrac{1}{2}\Lambda_1\pm[(a^2+b^2+c^2)/3]^{1/2}\sqrt{3}\sin\theta,$$

where

$$\theta=\tfrac{1}{6}\pi+\tfrac{1}{3}\sin^{-1}f\sin k,$$
$$f=abc[\tfrac{1}{3}(a^2+b^2+c^2)]^{-3/2}, \quad 0\le f\le 1. \quad (18)$$

The roots add up to zero at each k.—Note that $[\frac{1}{3}(a^2+b^2+c^2)]^{1/2}$ sets the overall scale and that the single variable f controls the rest. In these units, Λ_1 rises from 0 at $k=0$ to a value less than 1 at $k=\pi/2$ and drops to zero at $k=\pi$, while Λ_2 starts at $\sqrt{3}$, dips down to a value above 1 at $\pi/2$, and rises back to $\sqrt{3}$ at $k=\pi$ and $\Lambda_3=-(\Lambda_1+\Lambda_2)$. The curvature grows with f and at $f=1$ the two branches meet at $k=\pi/2$ with discontinuous slopes. This in turn leads to a singularity of the form $(1-f^2)^{-1/2}$ in $d^2\epsilon_0/df^2$, where ϵ_0 is the ground-state energy per site, in the natural units (due to the filled sea of λ_3 states)

$$\epsilon_0=-\int_0^\pi\sin[\tfrac{1}{3}\pi+\tfrac{1}{3}\sin^{-1}(f\sin k)]\pi^{-1}dk. \quad (19)$$

The physics of this model and the dependence on a, b, and c is reminiscent of Baxter's three-color problem.[4] The precise connection, if any, will be discussed elsewhere.

The O(8) Gross-Neveu model.[6]—The Lagrangian density is

$$\mathcal{L}=\tfrac{1}{2}\sum_{j=1}^{8}\bar{\psi}_j i\partial\!\!\!/\psi_j+g_0(\sum_{j=1}^{8}\bar{\psi}_j\psi_j)^2, \quad (20)$$

where $g_0>0$ and ψ_j is an O(8) isovector and Majorana spinor in 1+1 dimensions. Here is an overview. The discrete $\psi\to\gamma^5\psi$ symmetry gets spontaneously broken and the vacuum has $\langle\bar{\psi}\psi\rangle=\pm 4M$, where M is a dynamically generated mass set equal to unity hereafter. There are then kinks connecting positive ($\langle\bar{\psi}\psi\rangle>0$) and negative ($\langle\bar{\psi}\psi\rangle<0$) vacua. They come in two isomultiplets of eight each, R and L. It is possible to rewrite

$\mathcal{L}(\psi)$ in terms of R or L operators and one finds $\mathcal{L}(\psi)=\mathcal{L}(R)=\mathcal{L}(L)$. Since $\mathcal{L}$ is *invariant* we are at the point of self-triality.

To establish these claims I turn to the bosonic version of the model which serves as the meeting ground with the other two dual versions. I pair the eight Majorana fields into four Dirac fields $\Psi_1, \ldots, \Psi_4$ and bosonize the latter. For example, for $\Psi_1=(\psi_1+i\psi_2)/\sqrt{2}$,

$$\Psi_1 i\not\partial\Psi_1=\tfrac{1}{2}\psi_1 i\not\partial\psi_1+\tfrac{1}{2}\bar{\psi}_2 i\not\partial\psi_2=\tfrac{1}{2}(\partial_\mu\varphi_1)^2, \tag{21a}$$

$$J_1{}^\mu=\bar{\Psi}_1\gamma^\mu\Psi_1=\tfrac{1}{2}i(\bar{\psi}_1\gamma^\mu\psi_2-\bar{\psi}_2\gamma^\mu\psi_1) =\pi^{-1/2}\epsilon^{\mu\nu}\partial_\nu\varphi_1, \tag{21b}$$

$$\bar{\Psi}_1\Psi_1=\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2=\,:\cos[(4\pi)^{1/2}\varphi_1]:. \tag{21c}$$

The bosonized Lagrangian reads

$$\mathcal{L}=\tfrac{1}{2}\sum_{i=1}^{4}(\partial_\mu\varphi_i)^2+g_0\sum_{i=1}^{4}\sum_{\substack{j=1\\ j\neq i}}^{4}\{:\cos[(4\pi)^{1/2}\varphi_i]::\cos[(4\pi)^{1/2}\varphi_j]:\}. \tag{22}$$

For further details on this transformation see Witten[7] and Shanker.[8]

Examining the potential energy term, we see that the positive vacua correspond to $\varphi_i/\sqrt{\pi}=n_i$, $n_i=0,\pm 1,\pm 2,\ldots$, and the negative vacua to $\varphi_i/\sqrt{\pi}=n_i+\frac{1}{2}$. If $\varphi\to 0$ as $x\to-\infty$ is assumed, there are solitons that interpolate, as $x\to\infty$, to one of the eight positive vacua or one of the sixteen negative vacua nearest to $\varphi=0$. The former are just the four Ψ_i and their antiparticles, the latter are the isospinor kinks which are classified R or L accordingly as the number of coordinates in $\varphi/\sqrt{\pi}$ $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ is even or odd. To understand these multiplet assignments, one must integrate Eq. (21b) over x to get

$$\pi^{-1/2}[\varphi_i(\infty)-\varphi_i(-\infty)]=\int_{-\infty}^{\infty}J_i{}^0dx=H_i, \tag{23}$$

where H_i are the commuting O(8) generators of rotations in the 1-2, 3-4, 5-6, and 7-8 planes. Thus, $\varphi_i(\infty)/\sqrt{\pi}$ are the O(8) weights. Note that only the isospinors disorder $\langle\bar{\psi}\psi\rangle$ and deserve to be called kinks from the point of view of $\mathcal{L}(\psi)$.

Now O(8) is special[8] in that besides the weights of the vector representation, the R and L weights each defines an *orthonormal* set of basis vectors. Switching to the basis generated by R and calling η_i the coordinate measured along these directions, we see that R now has integral coordinates in $\eta_i/\sqrt{\pi}$ while ψ and L have half-integral coordinates. Thus $\varphi\leftrightarrow\eta$ exchanges ψ and R. To see how the R's interact, we rewrite $\mathcal{L}$ in terms of η and find[8] $\mathcal{L}(\varphi)=\mathcal{L}(\eta)$! In other words, if we fermionize $\mathcal{L}(\eta)$ we will get a Gross-Neveu model in R with a $g_0(\bar{R}R)^2$ interaction term. Likewise we can show that $\mathcal{L}(L)=\mathcal{L}(\psi)$ also. Since $\mathcal{L}$ is *invariant* under the triality transformation, we are at the self-triality point. We are yet to find that more general three-parameter theory with self-triality which yields the present model at the self-triality point.

The construction of models with self-duality (triality) and their fermionic solutions.—Recall how the horribly nonlocal change of variables $\psi\to R$ is effected by a simple local coordinate transformation in the bosonized versions of $L(\psi)$ and $L(R)$. I believe that all self-dual Hamiltonians will have an intermediate version in which the duality transformation is trivial. Conversely, self-dual models can be built by starting with the intermediate version and working backwards, which was how the spin problem discussed here was arrived at. To sharpen these ideas, let us note that the Ising Hamiltonian can be written in a *mixed* basis as

$$H=-\tfrac{1}{2}a\sum_{\substack{\text{dual}\\ \text{sites}}}\mu_1-\tfrac{1}{2}b\sum_{\text{sites}}\sigma_1, \tag{24}$$

with the requirement that $\mu_1{}^2=\sigma_1{}^2=1$ and that each μ_1 or σ_1 anticommute with its nearest dual neighbors and commute with all else. Given this symmetric definition, the self-duality of H is obvious and corresponds to $\mu_1\leftrightarrow\sigma_1$. Traditionally one satisfies the above-mentioned algebra by choosing μ_1 as $\sigma_3\sigma_3'$ and $\mu_3=\overrightarrow{\prod}\sigma_1$ [which not only makes μ isomorphic to σ, it also ensures $\mu(\sigma)=\sigma(\mu)$] and then works with either $H(\sigma_1,\sigma_3)$ or $H(\mu_1,\mu_3)$. But there exists a way of directly confronting $H(\sigma_1,\mu_1)$. Introduce two self-adjoint fermion variables ψ_1 and ψ_2 obeying

$$\{\psi_i(m),\psi_j(n)\}=\delta_{ij}\delta_{mn}. \tag{25}$$

Clearly choice of $\sigma_1=2i\psi_1\psi_2$ and $\mu_1=2i\psi_1\psi_2'$ does the job and we get

$$H=-a\sum i\psi_1\psi_2'-b\sum i\psi_1\psi_2. \tag{26}$$

Solving this quadratic form, one obtains the familiar result of Schultz, Mattis, and Lieb.[9]

For the self-triality model discussed here, imagine a lattice of γ_2's and two dual lattices displaced to the left (right) by a third of a lattice unit and carrying λ_2 (μ_2) variables. Demand that each variable anticommute with its nearest dual variables and commute with all else. Then

$$H = -\tfrac{1}{2}\sum (a\gamma_2 + b\mu_2 + c\lambda_2) \tag{27}$$

has self-triality. One then satisfies the anticommutation algebra either with $\mu_2 = \gamma_3$ and $\lambda_2 = \gamma_{34}\gamma_{12}'$ and deals with $H(\gamma)$ or one sets $\gamma_2 = 2i\psi_1\psi_2$, $\mu_2 = 2i\psi_1\psi_3$, and $\lambda_2 = 2i\psi_2\psi_3'$ and gets the quadratic form $H(\psi)$.[10] But note that in all these cases the existence of a fermion solution to the algebra is of no use unless we also have a solution in terms of σ or γ matrices, for the free Fermi theory is interesting only because it maps onto a spin problem.

Since completing this analysis, I have become aware of the work of Srednicki, Fradkin, and Suskind,[11] which also arrives at some of the notions I discussed towards the end of this work on the role of fermion variables in spin systems.

A longer paper discussing many more issues is in preparation.

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Note added.—I have been informed by Professor E. Lieb of related work by Bashilov and Pokrovsky.[12]

[1]See, for example, L. Susskind and E. Fradkin, Phys. Rev. D 17, 2637 (1978), and references therein.

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[9]T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. 36, 856 (1964). Here we will find two branches $\Lambda_{1,2}(k) = \pm a(1+\lambda^2+2\lambda\cos k)^{1/2}$, where $0 \leq k \leq \pi$ and $\lambda = b/a$. Filling up the Λ_2 sea, we get a Majorana particle with $\Lambda(k) = a(1+\lambda^2+2\lambda\cos k)^{1/2}$, $-\pi \leq k \leq \pi$, and a vacuum energy of $-\frac{1}{2}\Sigma_{-\pi}^{\pi}\Lambda(k)$.

[10]In completing the μ and λ sets, we must also attain the following symmetry in the transformation functions: $\gamma(\lambda)$ must be the same function as $\mu(\lambda)$ if H is to be invariant under triality. There exist other sets of variables isomorphic to γ that do not satisfy this.

[11]M. Srednicki, Phys. Rev. D 21, 2878 (1980). E. Fradkin, M. Srednicki, and L. Susskind, Phys. Rev. D 21, 2285 (1980), and references therein.

[12]Yu A. Bashilov and S. V. Pokrovsky, Landau Institute Report No. UD5-539.12, 1980 (to be published).

Black Holes Do Evaporate Thermally

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A careful consideration of the propagation of null geodesics in a black-hole geometry modified by Hawking-radiation back reaction shows that the event horizon is stable and shifted only very slightly in radius from what is expected in a vacuum background, contrary to a recent claim in the literature.

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The original derivation of the Hawking radiation from black holes[1,2] assumed that the black hole is static or stationary for the purpose of calculating the rate and properties of the radiation. Overall energy conservation requires that the quantum fluctuations in the vicinity of the event horizon responsible for the radiation produce an energy-momentum tensor whose expectation value corresponds to a negative energy flux into the black hole. Several attempts[2,3-6] have been made at a direct calculation of the effective energy-momentum tensor, which in the absence of a manageable quantum theory of gravity can be used in a semiclassical approximation to find the evolution of the black hole. The geometry, through the classical Einstein equations, and therefore the propagation of null geodesics is modified from that expected in a vacuum black-hole metric.

Tipler[7] recently considered this back reaction for spherically symmetric black holes and con-