results for $S(\vec{q})$ in Figs. 2 and 3 show how the power-law behavior grows out of the characteristic diffraction pattern of a finite lattice. The picture which emerges is quite different from that of a single Bragg peak of width $1/L$ combined with a power-law singularity, such as suggested by previous workers.[2,3]

The most important implication of our results is that one can expect to see the Landau-Peierls power-law behavior characteristic of the infinite lattice $S(\vec{q})$ as soon as one is dealing with crystallites of size $L \sim 5000$ Å. This seems especially significant in view of the recent report[17] of crystallites of just such a size in a new form of exfoliated graphite.

One of us (A.G.) would like to thank Dr. P. Dutta and Dr. S. K. Sinha for a useful discussion of their work. This work was supported by a grant from the National Research Council of Canada.

(a)Present address: Department of Physics, Louisiana State University, Baton Rouge, La. 70803.

(b)Address until September, 1981: Institut Laue-Langevin, 156X-38042 Grenoble, France.

[1]B. Jancovici, Phys. Rev. Lett. 19, 20 (1967).

[2]Y. Imry and L. Gunther, Phys. Rev. B 3, 3939 (1971); see also B. Gavish and Y. Imry, J. Chem. Phys. 65, 139 (1976).

[3]Y. Imry, Crit. Rev. Solid State Mater. Sci. 8, 157 (1979).

[4]J. P. McTague, M. Nielsen, and L. Passell, Crit. Rev. Solid State Mater. Sci. 8, 135 (1979).

[5]P. Vora, S. K. Sinha, and R. K. Crawford, Phys. Rev. Lett. 43, 704 (1979).

[6]R. J. Birgeneau, E. M. Hammonds, P. Heiney, P. W. Stephens, and P. M. Horn, in *Proceedings of the International Conference on Ordering in Two Dimensions, Lake Geneva, Wisconsin, 1980* (North-Holland, New York, 1980).

[7]*Theory of Lattice Dynamics in the Harmonic Approximation: Second Edition*, edited by A. A. Maradudin, E. W. Montroll, G. H. Weiss, and I. P. Ipatova (Academic, New York, 1971), p. 300 ff.

[8]L. Gunther, Phys. Lett. 25A, 649 (1967), and 26A, 216 (1968).

[9]D. R. Nelson and B. I. Halperin, Phys. Rev. B 19, 2457 (1979).

[10]We might remark that this is similar to the effect of the substrate potential on $S(\vec{q})$ in an infinite 2D commensurate lattice, insofar as it produces an energy gap in the phonon spectrum of the monolayer. For further discussion of substrate effects, see G. L. Cleveland, C. S. Brown, and U. Landman, Phys. Rev. B 22, 1284 (1980).

[11]J. G. Dash, *Films on Solid Surfaces* (Academic, New York, 1975), Chap. 7.

[12]F. Weling and A. Griffin, to be published.

[13]M. Bretz, Phys. Rev. Lett. 38, 501 (1977).

[14]We also have explicitly verified that our continuum approximation for the lattice sums makes negligible error in the region $q - K \lesssim 0.01$ Å^{-1}.

[15]See, for example, J. K. Kjems, L. Passell, H. Taub, J. G. Dash, and A. D. Novaco, Phys. Rev. B 13, 1446 (1976).

[16]P. Dutta, S. K. Sinha, P. Vora, M. Nielsen, L. Passell, and M. Bretz, in *Proceedings of the International Conference on Ordering in Two Dimensions, Lake Geneva, Wisconsin, 1980* (North-Holland, New York, 1980).

[17]D. E. Moncton, P. W. Stephens, R. J. Birgeneau, P. M. Horn, and G. E. Brown, to be published (see also Ref. 6).

Scales and Scaling in the Kondo Model

N. Andrei and J. H. Lowenstein
Department of Physics, New York University, New York, New York 10003
(Received 29 September 1980)

The thermodynamics of the Kondo model are formulated in terms of coupled integral equations and various properties, in particular the scaling property, are deduced. Then, with definition of the various scales parametrizing various asymptotic regions of the *H-T* plane, universal numbers are calculated and, in particular, Wilson's result is obtained analytically.

PACS numbers: 75.20.Hr

Recently it was shown[1] that the Kondo Hamiltonian[2] can be exactly diagonalized with use of Bethe-*Ansatz* techniques. It is our purpose in this note to extend the formulation to nonzero temperatures, showing how the phenomenon of scaling arises in the model. Then, by means of explicit perturbative and nonperturbative calculations, we shall determine the dimensional scales which parametrize the thermodynamic quantities in three asymptotic domains. The ratios of these scales are pure numbers which should be independent of cutoff procedure or definition of unrenormalized coupling constant. Hence it will be instructive to compare our results with those obtained by Wilson[3] with use of very different methods.

Our first task is to obtain a set of equations which determine the thermodynamics of a system of N^e electrons interacting with N^i impurities[4] on a line segment of length L. It was shown in Ref. 1 that the energy eigenvalues are given (dropping some constants) by

$$E=\sum_{j=1}^{N^e} 2\pi n_j L^{-1} + D\sum_{\gamma=1}^{M}[\theta(2\Lambda_\gamma - 2)-\pi], \tag{1}$$

where $D=N^e/L$ is the electron density, n_j are integers, and the set of M complex numbers Λ_γ satisfy

$$N^e\theta(2\Lambda_\gamma-2)+N^i\theta(2\Lambda_\gamma) = -2\pi I_\gamma + \sum_{\delta=1}^{M}\theta(\Lambda_\gamma-\Lambda_\delta),\quad \gamma=1,\ldots,M, \tag{2}$$

where $\theta(x)=-2\tan^{-1}(x/c)$, $-\pi\leq\theta<\pi$; $c=2J(1-\frac{3}{4}J^2)^{-1}$; J is a coupling constant, and the I_γ are integers (half-integers) when $N-M$ is odd (even).[5] The corresponding eigenstates have spin $S=\frac{1}{2}(N-2M)$ and $|S_z|\leq S$.

Equation (2) has the same structure as the one obtained in the solution of the Heisenberg model by Bethe-*Ansatz* techniques.[6] This equation was used there following a method devised by Yang and Yang[7] to construct a set of integral equations which determine the thermodynamics. Following the identical steps, we find

$$F=E_0+F^b-(T/2c)\int d\Lambda\, N^e \operatorname{sech}[\pi(\Lambda-1)/c]\ln[1+\eta_1(\Lambda)]-(T/2c)\int d\Lambda\, N^i \operatorname{sech}(\pi\Lambda/c)\ln[1+\eta_1(\Lambda)] \equiv E_0+F^b+F^e+F^i, \tag{3}$$

where E_0 is the ground-state energy and F^b is the free energy of a gas of noninteracting bosons, describing density excitations associated with the quantum numbers n_j. The function η_1 is the first member of a set of functions η_n satisfying

$$\ln\eta_1=-2DT^{-1}\tan^{-1}\{\exp[\pi(\Lambda-1)/c]\}+G\ln(1+\eta_2), \tag{4}$$

$$\ln\eta_n=G\ln(1+\eta_{n-1})+G\ln(1+\eta_{n+1}),\quad n>1, \tag{5}$$

subject to the boundary condition

$$[n]\ln(1+\eta_{n+1})-[n+1]\ln(1+\eta_n)\xrightarrow{n\to\infty}2H/T, \tag{6}$$

where $[n]$ and G are integral operators with kernels $\pi^{-1}(nc/2)[(nc/2)^2+\Lambda^2]^{-1}$ and $(2c)^{-1}\operatorname{sech}(\pi\Lambda/c)$, respectively.[8]

Many properties of the model may be deduced without an explicit solution of the equations. The most important one is the scaling property[9] (dimensional transmutation): In the scaling regime $T\ll D$, a scale is dynamically generated which completely characterizes the physics. In other words, there is no explicit dependence on D or J except through the combination $T_0=D\exp(-\pi/c)$.

To see this we note that as a consequence of (4), η_1 goes to zero for large Λ proportionally to $\exp(-2\times DT^{-1}\tan^{-1}z)$, where $z=\exp[\pi(\Lambda-1)/c)]$. Thus, if z does not satisfy $z\ll 1$, then η_1 is of order $\exp(-2\times D/T)$ and contributes negligibly to the integrals in Eqs. (3) and (5). Hence for $T\ll D$ we may replace $\tan^{-1}z$ by z and, rewriting the equations as functions of a new variable $\xi=\pi\Lambda/c+\ln(T_0/T)$, we have

$$\ln\eta_1=-2e^\xi+G\ln(1+\eta_2);\quad \ln\eta_n=G\ln(1+\eta_{n+1})+G\ln(1+\eta_{n-1}),\quad n>1; \tag{7}$$

and

$$F^i=N^i(T/2\pi)\int d\xi \operatorname{sech}[\xi-\ln(T_0/T)]\ln[1+\eta_1(\xi,H/T)]; \tag{8}$$

so that $F^i=-N^iTf^i(T/T_0,H/T)$ is a universal function.[9]

Consider now the low-temperature region $T\ll T_0$. As the integrand is sufficiently damped at large ξ we may expand the sech and find

$$F^i=-N^iT^2(\pi T_0)^{-1}\int d\xi\, e^\xi\ln[1+\eta_1(\xi,H/T)].$$

Since η_1 is a smooth function of H/T, the expression exhibits the Kondo effect, namely, that as the temperature is lowered, the impurity susceptibility $\chi_i=-\mu^2[\partial^2F^i/\partial H^2]_{H=0}$ remains finite, indicating a transition to a singlet ground state.

At the other end of the temperature scale, $T \gg T_0$ (but still $T \ll D$), only that portion of the ξ axis with ξ large and negative contributes appreciably to F^i. From this fact one can construct a high-temperature expansion of the free energy and susceptibility. In particular, we find (details will be published elsewhere)

$$\chi^i = N^i T^{-1}\mu^2\{1 - [\ln(T/T_0)]^{-1} - \tfrac{1}{2}[\ln(T/T_0)]^2 \ln\ln(T/T_0) + O([\ln(T/T_0)]^{-2})\}. \quad (9)$$

This result clarifies the connection between our definition of the cutoff[1] and the conventional scheme.[10] The high-temperature region is accessible by perturbation theory and the known result is

$$\chi^i = N^i T^{-1}\mu^2[1 - 2g/\pi + O(g^2)],$$

where g is the coupling constant defined conventionally, with momentum cutoff $\mathfrak{D}$, while by expanding Eq. (9) to second order in J we have

$$\chi^i = N^i T^{-1}\mu^2\left[1 - \frac{2J}{\pi} + \frac{1}{2}\left(\frac{2J}{\pi}\right)^2 \ln\left(\frac{2J}{\pi}\right) + O(J^2)\right].$$

Obviously the two cutoff schemes lead to a nonanalytic relation between the corresponding coupling constants

$$\frac{2g}{\pi} = \frac{2J}{\pi} - \frac{1}{2}\left(\frac{2J}{\pi}\right)^2 \ln\frac{2J}{\pi} + \cdots .$$

This nonanalyticity is the origin of the nonuniversality of the second term in the β function and accounts for our expression for the scale, $T_0 = D\exp(-\pi/c)$, as compared with the conventional expression, $\mathfrak{D}\exp(-\pi/2g + \frac{1}{2}\ln g + \cdots)$.[10] Indeed, our cutoff procedure differs from the conventional one in that it allows discontinuities in the basis wave functions. Still, we expect that in the scaling region cutoff effects can be neglected. Different constructions of the model may lead to different dependence of T_0 on the bare parameters; the dependence of physical quantities on T_0 is, however, universal. Our calculations in this model (see below) and in related models[11] provide ample evidence in support of this assertion, though a complete proof of equivalence is still lacking.

We turn now to the discussion of various scales that characterize the behavior of the magnetization $\mathfrak{M}^i$ in the asymptotic regions (I) $T=0$, $H \ll T_0$; (II) $T=0$, $H \gg T_0$; and (III) $T \gg T_0$, $H \ll T_0$. We denote them by T_0, T_H, and T_K (the Kondo temperature, by convention), respectively. To be precise

$$\mathfrak{M}^i(\mathrm{I}) \sim \mu\pi^{-1}H/T_0;$$

$$\mathfrak{M}^i(\mathrm{II}) \sim \mu\{1 - \tfrac{1}{2}[\ln(H/T_H)]^{-1} - \tfrac{1}{4}[\ln(H/T_H)]^{-2}\ln\ln(H/T_H) + O([\ln(H/T_H)]^{-3})\};$$

$$\mathfrak{M}^i(\mathrm{III}) \sim \mu(H/T)\{1 - [\ln(T/T_K)]^{-1} - \tfrac{1}{2}[\ln(T/T_K)]^{-2}\ln\ln(T/T_K) + O([\ln(T/T_K)]^{-3})\}.$$

The expression for $\mathfrak{M}^i(\mathrm{III})$ corresponds to Eq. (9) above; the other two will be deduced below.

As renormalized quantities, the dimensional scales T_0, T_H, T_K are not individually calculable. However, their ratios are universal numbers independent of how the ultraviolet cutoff and unrenormalized coupling constant are defined in a particular construction of the model. One of the ratios, T_K/T_H, may be calculated in perturbation theory, whose domain of validity includes regions (II) and (III). Either of the other ratios, however, relates quantities measured far below and far above the Kondo temperature and hence requires a nonperturbative treatment of the crossover. In his classic paper[3] Wilson applies renormalization-group techniques and a good deal of computer work to bridge the gap. Our formalism allows a much simpler approach: The formulation of the zero-temperature magnetization equation in Ref. 1 allows one to compute explicitly the magnetization for all values of H and thus obtain the ratio T_H/T_0. Combining it with T_K/T_H then allows comparison with Wilson's value for $W = T_K/T_0$.

The ratio T_K/T_H is computed by evaluating the free energy to second order in perturbation theory. One finds

$$\frac{F^i}{T} = -\ln\left[2\cosh\left(\frac{H}{T}\right)\right] + [H\text{-independent terms (H.I.T.)}] + \frac{g}{\pi}\frac{H}{T}\tanh\frac{H}{T}$$
$$-\left(\frac{g}{\pi}\right)^2\left[\frac{1}{2}\frac{H^2}{T^2}\operatorname{sech}^2\frac{H}{T} + \left(\frac{\exp(H/T)}{2\cosh(H/T)}\left\{\varphi\left(\frac{H}{T}\right) - \frac{2H}{T}\left[1 + \ln\left|\frac{\mathfrak{D}}{2H}\right| + \mathrm{Ei}\left(-\frac{2H}{T}\right)\right]\right\} + H \to -H\right)\right] + O(g^3),$$

where

$$\varphi(t) = \int_0^1 dx\,(1-x)\exp(-2xt)[\pi^2\csc^2(\pi x) - x^{-2} - (1-x)^{-2}],$$

and Ei is the exponential-integral function. Asymptotically,

$$F^i \sim -H + (\text{H.I.T.}) + gH/\pi + 2H(g/\pi)^2\ln(e\mathfrak{D}/2H) + O(g^3), \quad T/H \to 0;$$

$$F^i \sim -H^2/2T + (\text{H.I.T.}) + gH^2/\pi T + 2(H^2/T)(g/\pi)^2\ln(\beta\gamma e^{-7/4}\mathfrak{D}/T) + O(g^3), \quad H/T \to 0;$$

where

$$\ln\beta = \int_0^1 dx\,(1-x)^2 x(\pi^2\csc^2\pi x - x^{-2})$$

and $\ln\gamma$ is Euler's constant. Thus

$$T_K/T_H = 2\beta\gamma e^{-7/4}.$$

We proceed now to calculate T_H/T_0. The method was outlined and used in Ref. 1. It was shown that the lowest-energy state at $T=0$ and $H \neq 0$ is characterized by having only real Λ's with a density function σ computed from the equation

$$\sigma(\Lambda) + \int_B^\infty K(\Lambda - \Lambda')\sigma(\Lambda')d\Lambda' = f(\Lambda), \tag{10}$$

where

$$K(x) = c\pi^{-1}(c^2+x^2)^{-1}, \quad f(x) = 2c\pi^{-1}\{N^e[c^2+4(x-1)^2]^{-1} + N^i(c^2+4x^2)^{-1}\}.$$

The constant B is determined by the field H and describes the response of the ground state to the applied field.

Written in terms of $\rho(\Lambda) = \sigma(\Lambda + B)$, Eq. (10) is of the Wiener-Hopf type[12] and the method of solution was given by Yang and Yang.[13] One factorizes the Fourier-transformed kernel,

$$1 + \tilde{K}(p) = 2\exp(-\tfrac{1}{2}c|p|)\cosh(\tfrac{1}{2}cp) = K_+(cp/2\pi)/K_-(cp/2\pi),$$

$$K_+(q) = K_-(-q)^{-1} = (2\pi)^{1/2}\exp\{-iq[1 + i\pi/2 - \ln(-q+i0)]\}/\Gamma(\tfrac{1}{2}+iq),$$

to transform Eq. (10) into the following equation for the Fourier transforms of $\rho_\pm(\Lambda) = \theta(\pm\Lambda)\rho(\Lambda)$:

$$K_-(cp/2\pi)\tilde{\rho}_-(p) + K_+(cp/2\pi)\tilde{\rho}_+(p) = K_-(cp/2\pi)\tilde{f}(p)\exp(ipB) = K_+(cp/2\pi)\tilde{g}(p)\exp(ipB),$$

where

$$g(\Lambda) = (2c)^{-1}\{N^e\operatorname{sech}[\pi(\Lambda-1)/c] + N^i\operatorname{sech}(\pi\Lambda/c)\}.$$

Laplace transforming $f(\Lambda)$ and $g(\Lambda)$, one can solve Eq. (10) to obtain $\tilde{\rho}_\pm(p)$, and hence the magnetization

$$\mathfrak{M} = \mu\int_{-\infty}^B \sigma(\Lambda)d\Lambda = \mu\tilde{\rho}_-(0) = \mu[N - 2\tilde{\rho}_+(0)] = \mathfrak{M}^e + \mathfrak{M}^i,$$

where[14] (again we consider only $H \ll D$)

$$\mathfrak{M}^e = \mu(2/\pi e)^{1/2}LT_0\exp(\pi B/c), \quad B \ll 1; \tag{11}$$

$$\mathfrak{M}^i = \begin{cases} N^i\mu\pi^{-1/2}\sum_{k=0}(-1)^k(k!)^{-1}(k+\tfrac{1}{2})^{k-1/2}\exp[-(k+\tfrac{1}{2})][\exp(\pi B/c)]^{2k+1}, & B \leq 0 \\ N^i\mu\{1 - \pi^{-3/2}\int_0^\infty dt\,t^{-1}\sin(\pi t)\exp(-2\pi Bt/c)\exp[-t(\ln t - 1)]\Gamma(t+\tfrac{1}{2})\}, & 0 \leq B \ll 1. \end{cases} \tag{12}$$

Setting $N^i = 0$, we have a free-electron system so that

$$\mathfrak{M}^e = \mu LH/\pi = \mu(2/\pi e)^{1/2}e^{\pi B/c}LT_0,$$

which allows us to identify $e^{\pi B/c} = (e/2\pi)^{1/2}H/T_0$. Note that the condition $B \ll 1$ is equivalent to $H \ll D$ and that we again have a scaling phenomenon, namely the impurity magnetization $\mathfrak{M}^i$ depends only on H/T_0.

In the limit $H \to 0$ we find $\mathfrak{M}^i \to \mu N^i H/\pi T_0$, consistent with our definition of T_0, while for $H \gg T_0$ (still $H \ll D$) we find

$$\mathfrak{M}^i = \mu N^i\{1 - \tfrac{1}{2}[\ln(H/T_H)]^{-1} - \tfrac{1}{4}[\ln(H/T_H)]^{-2}\ln\ln(H/T_H) + O([\ln(H/T_H)]^{-3})\},$$

where

$$T_H/T_0 = (\pi/e)^{1/2}.$$

Note that again if we expand the result in the coupling constant we find the same nonanalytic relation between J and the conventionally defined coupling constant g.

Now from the two universal numbers T_K/T_H and T_H/T_0 we may deduce a third one:

$$W = \frac{T_K}{T_0} = \frac{T_K}{T_H}\frac{T_H}{T_0} = 2\beta\gamma\pi^{1/2}e^{-9/4},$$

which appears in the famous relation[3]

$$\chi^i = W\pi^{-1}\mu^2/T_K,$$

and we find, substituting the values $\ln\gamma = 0.577\,216$ and $\ln\beta = 0.662\,122$, that $W/4\pi = 0.102\,676$ in agreement with Wilson's numerical answer $W/4\pi = 0.1032 \pm 0.0005$.

We would like to thank E. Witten and L. Yaffe for illuminating discussions and M. Croft for a timely question. We are grateful to J. Sak and K. G. Wilson for useful criticism and to P. W. Anderson for his support.

One of us (J.H.L.) did some of the research reported on in this article while a summer visitor at Brookhaven National Laboratory. This work was supported in part by the National Science Foundation under Grant No. PHY-78-21503.

[1]N. Andrei, Phys. Rev. Lett. 45, 379 (1980). See also P. B. Wiegmann, Pis'ma Zh. Eksp. Teor. Fiz. 31, 392 (1980) (JETP Lett., to be published).

[2]J. Kondo, Prog. Theor. Phys. 32, 37 (1964).

[3]K. G. Wilson, Rev. Mod. Phys. 67, 773 (1975).

[4]Note that the impurities have no three-dimensional analogs for $N_i > 1$ and that the electrons move in one direction only.

[5]Note that the formula for c given in Ref. 1 was erroneous. We thank those readers who called it to our attention.

[6]M. Gaudin, Phys. Rev. Lett. 26, 1301 (1971); M. Takahashi, Prog. Theor. Phys. 46, 401, 1388 (1971).

[7]C. N. Yang and C. P. Yang, J. Math. Phys. 10, 1115 (1969).

[8]The magnetic field is expressed as a multiple of μ, the magnetic moment of a single electron or impurity. Note that our definition of μ differs by a factor of 2 from that of Ref. 3.

[9]P. W. Anderson, G. Yuval, and D. R. Hamann, Phys. Rev. B 1, 4464 (1970).

[10]See Ref. 3 and references therein, e.g., M. Fowler and A. Zawadowski, Solid State Commun. 9, 471 (1971).

[11]N. Andrei and J. H. Lowenstein, Phys. Rev. Lett. 43, 1698 (1979), and Phys. Lett. 90B, 106 (1980), and 91B, 401 (1980).

[12]P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 8.

[13]C. N. Yang and C. P. Yang, Phys. Rev. 150, 327 (1966).

[14]Expression (12) was independently derived by Weigmann, Ref. 1, and to be published.

Observation of a Non-Ohmic Hall Resistivity at Low Temperatures in a Two-Dimensional Electron Gas

D. J. Bishop, D. C. Tsui, and R. C. Dynes
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 24 July 1980)

Hall measurements of electron inversion layers at low temperatures and electric fields are reported. For resistances of $\lesssim 10$ kΩ/□ a logarithmic dependence of the Hall coefficient on temperature and Hall voltage is observed. This indicates that the logarithmic dependences observed for the resistance of metal films and metal-oxide-semiconductor field-effect transistors are not evidence for the scaling theory of localization. These results are also difficult to resolve within the theory of interaction effects by Altshuler, Aronov, and Lee.

PACS numbers: 73.40.Qv, 73.25.Ti

Several experiments have recently observed logarithmic dependences of the resistivity on temperature and electric field which scale with sheet resistance $R_\square$ for two-dimensional (2D) systems at low temperatures.[1,2] In addition, for silicon inversion layers, Kawaguchi and Kawaji have reported a negative magnetoresistance.[3] Although these observations qualitatively agree with the basic predictions of the scaling theory of localization by Abrahams, Anderson, Licciardello, and Ramakrishnan (AALR)[4] there is now an alternative theory by Altshuler, Aronov, and Lee

