Temperature Dependence of the Condensate Fraction in Superfluid ^{4}He

Moorad Alexanian[(a)]
Departamento de Física, Centro de Investigación y de Estudios Avanzados, México 14, D.F., Mexico
(Received 22 April 1980)

A temperature-dependent expression for the condensate fraction n_0 for He II based on a model Hamiltonian for superfluids is presented. This gives roughly the same magnitude for n_0 as is observed. The model also suggests the following behavior as $T \to 0$ for the roton effective mass $\mu_r(T) = \mu_r(0)[1-(m_{\rm He}/24\hbar^3 cn)(k_BT)^2]$ and the roton minimum energy $\Delta(T) = \Delta(0)[1-(m_{\rm He}/24\hbar^3 cn)(k_BT)^2]$.

PACS numbers: 67.40.-w

Superfluidity[1] in helium is associated with the presence below the λ point of a "condensate." Now the simplest form of condensation is that occurring in the ideal Bose gas. Thus, a natural assumption of early theories of superfluidity was the existence of a finite macroscopic fraction n_0 of the particles in the zero-momentum state. However, no successful microscopic theory—comparable with the BCS theory—has been developed along this line notwithstanding the many efforts, and this has lead some people to consider more complicated condensates.[2,3] Thus, the model for He II of Ref. 2 features macroscopic occupation of infinitely many single-particle momentum states in an arbitrarily small neighborhood of the origin $k^2=0$; and so possesses a nonuniform condensate. This new model is a generalization of Bogoliubov's model and gives rise to not only a phonon spectrum, superfluidity, and a phase transition, but also to a roton spectrum, a condensate fraction of 10%, and the correct temperature and density dependence of the roton parameters and speed of first sound. Now rigorous proofs[3] demand that a Bose gas in spatial dimensionality $d \leq 2$ must possess a nonuniform condensate. Therefore, the model of Ref. 2 is applicable to two-dimensional (2D) superfluids and leads[3] to phononlike and rotonlike excitations for such Bose systems. Recent neutron-scattering work[4] further confirms the existence of a 2D roton with the properties given by the roton of Ref. 3. Consequently, the nature of the condensate is all important and has brought about an active experimental search for it and its properties; in particular, the temperature dependence of the condensate fraction.[5-7] In Ref. 2, the condensate fraction n_0 in superfluid ^{4}He was found to be related to the roton parameters—effective mass μ_r and momentum p_0—and the s-wave scattering length a in vacuum by

$$n_0 = \frac{3}{32\pi}\frac{p_0^2}{a\hbar^2 n}\left[\left(1+\frac{32}{9}\frac{\mu_r^2}{m_{\rm He}^2}\right)^{1/2}-1\right], \tag{1}$$

where n is the number density. Now[8] $p_0(\rho)/\hbar = A\rho^{1/3}$, with $A = 3.64$ cm g$^{-1/3}$ Å^{-1} and $a = 2.2$ Å, so that

$$n_0 \approx 2.14\,\rho^{-1/3}\,\mu_r^2/m_{\rm He}^2, \tag{2}$$

with ρ in grams per cubic centimeter. Therefore, the temperature dependence of the condensate fraction is directly related to the temperature dependence of μ_r. Thus, for $\rho = 0.143$ g/cm^3 and since[8] $\mu_r(1.26\,^\circ\mathrm{K}) = 0.160 m_{\rm He}$ at 1 atm, (2) becomes

$$n_0(T) = 0.105\,\mu_r^2(T)/\mu_r^2(1.26\,^\circ\mathrm{K}). \tag{3}$$

In Table I are listed the values for $n_0(T)$ as determined by (3) with use of the results for $\mu_r(T)$ along the $P = 1$ atm isobar [Ref. 8, Fig. 10, and Table I.] The values of $n_0(T)$ for $T \leq T_\lambda$ are shown by the filled triangles in Fig. 1. The solid curve is the result of a least-squares fit[7] of the relation

$$n_0(T) = n_0(0)[1-(T/T_\lambda)^\alpha] \tag{4}$$

to all the experimental values. One finds[7] with $T_\lambda = 2.17\,^\circ$K that $n_0(0) = 0.129 \pm 0.016$ and $\alpha = 5.4 \pm 3.8$. It is evident that our low-pressure results are quite consistent with the data at saturated vapor pressure.[5-7] However, the experimental error bars are at present too large to test very accurately the temperature dependence of our expression (3). Nevertheless, our result (3) lends

TABLE I. Values of the condensate fraction $n_0(T)$ as determined from Eq. (3).

T (K)	n_0
1.26	0.105 ± 0.007
1.45	0.109 ± 0.007
1.68	0.098 ± 0.007
2.11	0.078 ± 0.012
2.13	0.065 ± 0.006
2.17	0.058 ± 0.014

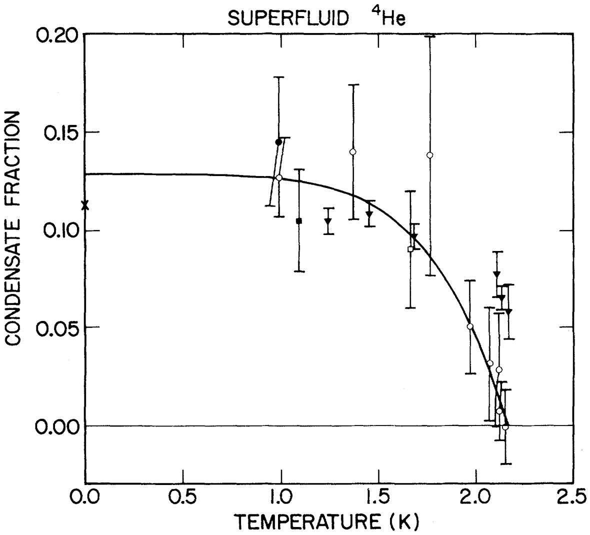

FIG. 1. The condensate fraction in superfluid ^{4}He. The open symbols show the values obtained from the temperature variation of the pair-correlation function: square, Ref. 5; circles, Refs. 6 and 7. The filled square and circle are from neutron inelastic-scattering measurements (Ref. 9). The filled triangles are determined from Eq. (3) and the × is from Monte Carlo (Ref. 10) and variational (Ref. 11) calculations at $T=0$. The curve represents a least-squares fit of Eq. (4) to all the experimental values.

further support to the assumption that $f(T) = (1-n_0)^2$ in the experimental observations[6,7] for the static pair correlation function $g(r,T)$ for $T < T_\lambda = 2.17\ {}^\circ\mathrm{K}$ and $r \gtrsim 6$ Å, which gives

$$g(r,T)-1 = f(T)[g(r, 2.27\ {}^\circ\mathrm{K})-1] \qquad (5)$$

with $f(T)$ decreasing monotonically with decreasing T. Note that the formal density-matrix arguments which yield $f(T) = (1-n_0)^2$ are not without criticisms.[7] The tentative nature of the $n_0(T)$ "measurements" should be emphasized since the experimental results reported in Refs. 5–7 constitute a measure of $n_0(T)$ only if the prescription $f(T) = (1-n_0)^2$ is in fact correct for the case of ^{4}He. Note also that an alternate explanation of the enhancement of the short-range spatial order can be based, interestingly enough, on the thermal excitation of rotons.[12] Since this theory[2] is based on a model Hamiltonian which neglects interactions between excitations, the results should be good up to about[1] 1.8 °K when the excitation density is sufficiently small. At temperatures in the neighborhood of the λ point, where the density of excitations would be very high, these results should not be applicable. For instance, it does not appear[8] that μ_r vanishes at T_λ as would be required by (3) and the expectation[13] that $n_0(T) \propto (T_\lambda - T)^{2\beta}$ near T_λ. It is interesting that for fixed T, μ_r decreases with increasing density[8]; and thus, by (2), n_0 is a decreasing function of density.

A rigorous microscopic derivation[14] of $n_0(T)$ as $T\to 0$ gives that $n_0(T) = n_0[1-(m_{\mathrm{He}}/12\hbar^3 cn)(k_B T)^2]$, where c is the velocity of sound at $T=0$. The T^2 law is based on exact result[15] for the one-particle Green's function in the limit of small k and ω at $T=0$. Unfortunately, these exact results are not obviously valid for Bose systems with nonuniform condensates; and, hence, we cannot conclude from (3) that rigorously

$$\mu_r(T) = \mu_r(0)[1-(m_{\mathrm{He}}/24\hbar^3 cn)(k_B T)^2] \qquad (6)$$

as $T\to 0$. Since[2] $\mu_r = m_{\mathrm{He}}{}^2\Delta/p_0{}^2$, a result analogous to (6) follows for the roton energy Δ. However, one can still obtain the T^2 law (6) for $\mu_r(T)$ by adopting a phenomenological point of view[16] where the T^2 law for $n_0(T)$ is valid as long as the low-energy excitations of the superfluid system are described by phonons with a linear dispersion law. Since the model of Ref. 2 gives rise to such phonons, the only caveat would be the macroscopic occupation of infinitely many single-particle quantum states with an accumulation at $\vec{k}=0$. In the expression for the static order-order correlation function $\langle\hat\psi^\dagger(\vec{x}')\hat\psi(\vec{x})\rangle$, we assume[16] that the moduli of the field operators are uncorrelated. Therefore,

$$\langle\hat\psi^\dagger(\vec{x}')\hat\psi(\vec{x})\rangle = \langle|\hat\psi(\vec{x}')|\rangle\langle|\hat\psi(\vec{x})|\rangle\langle\exp[-i\varphi(\vec{x}')]\exp[i\varphi(\vec{x})]\rangle, \qquad (7)$$

where $\varphi(\vec{x})$ is the phase of $\hat\psi(\vec{x})$. Thus, $\langle|\hat\psi(\vec{x})|\rangle = [n(\vec{x})]^{1/2}$, with

$$\langle\hat\psi^\dagger(\vec{x}')\hat\psi(\vec{x})\rangle = \Psi^*(\vec{x}')\Psi(\vec{x}) + (1/V)\sum_{\vec{k}} n_{\vec{k}}\exp[-i\vec{k}\cdot(\vec{x}'-\vec{x})] \qquad (8)$$

and $n(\vec{x}) = \langle\hat\psi^\dagger(\vec{x})\hat\psi(\vec{x})\rangle$, where $\Psi(\vec{x})$ is the condensate wave function and $n_{\vec{k}}$ is the helium-atom momentum distribution. Asymptotically, when $r \equiv |\vec{x}'-\vec{x}| \to \infty$ and $\vec{x}$, say, is of macroscopic order, $\langle\hat\psi^\dagger(\vec{x}')\hat\psi(\vec{x})\rangle = (N_0/V)\xi_0{}^2$, where N_0 is the total number of particles in the condensate and $\xi_0{}^2$ is the fraction of particles in the condensate with zero momentum. From (7) we have[16]

$$\langle \hat{\psi}^\dagger(\vec{x}\,')\hat{\psi}(\vec{x})\rangle = [n(\vec{x}\,')n(\vec{x})]^{1/2}\exp[F(r)-F(0)], \tag{9}$$

where

$$F(r) = (2\pi)^{-3}\int \frac{cm_{\rm He}}{n\hbar k}\left[\frac{1}{\exp(c\hbar k/k_B T)-1}+\frac{1}{2}\right]e^{i\vec{k}\cdot\vec{r}}d^3k. \tag{10}$$

Therefore, asymptotically as $r \equiv |\vec{x}\,' - \vec{x}| \to \infty$ and $\vec{x}$ is of macroscopic order, $\langle \hat{\psi}^\dagger(\vec{x}\,')\hat{\psi}(\vec{x})\rangle = (N/V)e^{-F(0)}$; and so, $(N_0/V)\xi_0^2 = (N/V)e^{-F(0)}$. Consequently, if ξ_0^2 is temperature independent, then $N_0(T)/N_0(0) = \exp(-\{F(0) - [F(0)]_{T=0}\})$ and, thus at low temperatures we have the T^2 law for $n_0(T)$. The temperature independence of ξ_0^2 is obtained in the nonuniform condensate model by showing that $R = \xi_0^2$, where R is the temperature-independent "renormalization constant" of the phenomenological approach.[16] Therefore, one obtains an understanding of the physical meaning of the constant R. Now the phenomenological approach[16] is concerned with a condensate composed of macroscopic occupation in the zero-momentum state only. Firstly, it gives[16] that $\lim_{k\to 0} n_k = (n_0/n) \times cm_{\rm He}/2\hbar k$ which is consistent with the microscopic[15] theory of a homogeneous system at zero temperature. Secondly, it gives for $T > 0$ that $\lim_{k\to 0} n_k = (n_0/n)m_{\rm He}k_B T/\hbar^2 k^2$ which is compatible with Bogoliubov's inequality for n_k; however, Bogoliubov's $1/k^2$ theorem does not necessarily follow[3] for spatially inhomogeneous condensates. Now for the unrenormalized quantities one has that $(N/V)e^{-F(0)} = (N_0/V)_u(\xi_0^2)_u$ and since[16] $(N_0/V)_u = (N/V)e^{-F(0)}$ we have that $(\xi_0^2)_u = 1$. In the model of Ref. 2, the number of particles in the condensate with zero momentum is $N_0\xi_0^2$ and in the absence of rotons $\xi_0^2 = 1$—in the limit $\xi_0^2 \to 1$ our model reduces to that of Bogoliubov with a uniform condensate and no rotons. Now the introduction[16] of the temperature-independent "renormalization constant" R takes into account the rotons which are difficult to calculate in a phenomenological manner. Therefore, $(N/V)e^{-F(0)}R = (N_0/V)_u(\xi_0^2)_u R$; hence, since $\xi_0^2 \equiv (\xi_0^2)_u R$, one has that $\xi_0^2 = R$. Consequently, $N_0(T)/N \equiv (N_0/N)_u R = e^{-F(0)}\xi_0^2$ which also leads to the T^2 law for $n_0(T)$. That is to say, the phenomenological approach is applicable to the zero-momentum occupation only but by introducing a renormalization factor and, thus, rotons, the condensate fraction is made to agree numerically with that for real superfluid helium. Notice that the numerical example[16] with a phonon cutoff of $mc/2\hbar = 0.755$ Å^{-1} gives $\exp\{-[F(0)]_{T=0}\} = 0.61$ and since from our model[2] $N_0(0)/N = 0.10$, we have that $\xi_0^2 = 0.16$. Therefore, this numerical example implies that of the 10% of particles in the condensate, an estimated 1.6% are in the zero-momentum state and this value is temperature independent up to about 1.8 °K.

The author wishes to thank Dr. V. F. Sears and Professor R. B. Hallock for useful correspondence.

[a]Presently on leave at the Physics Department, Montana State University, Bozeman, Mont. 59717.

[1]W. F. Vinen, in *Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1969), Vol. 2.

[2]M. Alexanian and R. A. Brito, Phys. Rev. B 17, 3547 (1978).

[3]M. Alexanian, Physica (Utrecht) 100A, 45 (1980), and references therein.

[4]W. Thomlinson, J. A. Tarvin, and L. Passell, Phys. Rev. Lett. 44, 266 (1980).

[5]H. N. Robkoff, D. A. Ewen, and R. B. Hallock, Phys. Rev. Lett. 43, 2006 (1979).

[6]V. F. Sears and E. C. Svensson, Phys. Rev. Lett. 43, 2009 (1979).

[7]V. F. Sears and E. C. Svensson, in Proceedings of the International Symposium on Fundamentals of Quantum Statistics and Quantum Theory, Palm Coast, Florida, March 1980 (to be published).

[8]O. W. Dietrich, E. H. Graf, C. H. Huang, and L. Passell, Phys. Rev. A 5, 1377 (1972).

[9]V. F. Sears, E. C. Svensson, P. Martel, and A. D. B. Woods, to be published.

[10]P. A. Whitlock, D. M. Ceperley, G. V. Chester, and M. H. Kalos, Phys. Rev. B 19, 5598 (1979).

[11]P. M. Lam and M. L. Ristig, Phys. Rev. B 20, 1960 (1979).

[12]C. De Michelis, G. L. Masserini, and L. Reatto, Phys. Lett. 66A, 484 (1978).

[13]B. D. Josephson, Phys. Lett. 21, 608 (1966).

[14]K. Kehr, Z. Phys. 221, 291 (1969).

[15]J. Gavoret and P. Nozières, Ann. Phys. (N.Y.) 28, 349 (1964).

[16]R. A. Ferrell, N. Menyhard, H. Schmidt, F. Schwabl, and P. Szepfalusy, Ann. Phys. (N.Y.) 47, 565 (1968).