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Two Distinct Local Potentials with No Bound States Can Have the Same Scattering Operator: A Nonuniqueness in Inverse Spectral Transformations

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By giving an explicit example in one dimension, it is shown that by relaxing a condition on the ranges, at least two potentials can have the same scattering operator in some cases. The two potentials are local and do not support point eigenvalues. The implications for solutions of the Korteweg–de Vries equation are briefly discussed.

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One of the principal assumptions in scattering experiments is that such experiments will yield the scattering potential. Hence it is important to know the kinds of lack of uniqueness in the potential recovered from the scattering data. In the present note we give an explicit example of two *local* potentials in the one-dimensional scattering problem, neither of which supports point eigenvalues, which have the same scattering operator. The present example is a new lack of uniqueness, which differs from two previous types, the first of which (Ref. 1) is due to a lack of complete specification of the spectral measure function of the discrete spectrum, and the second of which (Ref. 2) is due to the possibility of having nonlocal potentials with the same scattering properties. In the present example the two potentials are local and have no point spectrum so that the two previous ambiguities play no role.

Consider the one-dimensional Schrödinger equation for continuous spectrum eigenfunctions,

$$[-d^2/dx^2+V(x)]\psi(x|k)=k^2\psi(x|k) \qquad (-\infty<x<\infty). \tag{1}$$

We require the outgoing wave solutions of Eq. (1) which are denoted by $\psi(x|k,\alpha)$, with $k>0$ and $\alpha=\pm$. They satisfy the boundary conditions

$$\begin{aligned}
&\lim_{x\to-\infty}\psi(x|k,-)=e^{ikx}+b(k)e^{-ikx},\\
&\lim_{x\to+\infty}\psi(x|k,-)=t(k)e^{ikx},\\
&\lim_{x\to+\infty}\psi(x|k,+)=e^{-ikx}+c(k)e^{ikx},\\
&\lim_{x\to-\infty}\psi(x|k,+)=s(k)e^{-ikx}.
\end{aligned} \tag{2}$$

Clearly α gives the direction of the incoming wave. The quantities $b(k)$ and $t(k)$ are called the reflection and transmission coefficients, respectively, from the left and $c(k)$ and $s(k)$ are the corresponding quantities from the right. It is shown in many places (see Ref. 3, for example) that $s(k)=t(k)$ and that the matrix

$$S(k)=\begin{pmatrix} t(k) & c(k) \\ b(k) & t(k) \end{pmatrix}, \tag{3}$$

which is called the scattering operator, is unitary.

The principal point of the present note is to give two potentials $V_1(x)$ and $V_2(x)$ which have the same scattering operator. Let the potential $V_1(x)$ be given by

$$V_1(x)=-2\delta(x)+8\eta(x)\frac{1}{(2x+1)^2}, \tag{4}$$

where $\eta(x)$ is the Heaviside function, $\eta(x)=1$ for $x>0$ and $\eta(x)=0$ for $x\leq 0$. It is readily seen that the function $\psi(x|k)$ satisfies the Schrödinger equation (1) with $V(x)=V_1(x)$, where

$$\psi(x|k)=e^{ikx}+\eta(x)\frac{1}{2x+1}\times\left(-\frac{4ix}{k}\cos kx+\frac{4i}{k^2}\sin kx-\frac{2}{k}\sin kx\right). \tag{5a}$$

The complex conjugate of this wave function is also a solution of Eq. (1). Hence the outgoing wave function which satisfies the boundary conditions of Eq. (2) is a linear combination of $\psi(x|k)$ and $\psi^*(x|k)$. It is readily found that

$$b(k)=c(k)=\frac{i}{k+i}, \quad t(k)=\frac{k}{k+i}. \tag{5b}$$

As the second potential $V_2(x)$ we take

$$V_2(x)=V_1(-x) \tag{6}$$

for which the wave function analogous to $\psi(x|k)$ of Eq. (5a) is $\varphi(x|k)$ and is given by

$$\varphi(x|k)=\psi^*(-x|k). \tag{7}$$

By taking linear combinations as before, one obtains the scattering coefficients of Eq. (5b).

In Refs. 3 and 4 the direct and inverse problem of one-dimensional scattering is discussed. Two theorems of interest in the present discussion state that, for potentials $V(x)$ such that

$$\int_{-\infty}^{+\infty}|V(x)|(1+|x|)dx<\infty, \tag{8}$$

(1) if the reflection coefficient is such that $|b(0)|=1$, then $b(0)=-1$; (2) if there are no point eigenvalues, $b(k)$ determines $V(x)$ uniquely. One might think that condition Eq. (8) could be weakened and that the uniqueness theorem might still hold. To break condition (8) we chose a $b(k)$ such that $b(0)=1$, namely that of Eq. (5b). We then used the inverse scattering method of Ref. 5 with the Gelfand-Levitan equation from the left to obtain $V_1(x)$. This potential violates the condition Eq. (8) because of Theorem 1, though strictly speaking $|V_1(x)|$ can be given only in a heuristic sense because of the presence of the δ function, i.e., $|V_1(x)|=2\delta(x)+8\eta(x)(2x+1)^{-2}$. The condition Eq. (8) fails, not because of the δ function in $V_1(x)$, but because $V_1(x)$ has a long tail and the integral of Eq. (8) diverges logarithmically for large x. A similar situation holds for other examples to be given in later papers where no δ function appears and $|V(x)|$ has a proper meaning. [One of our objectives was to see whether a meaningful $V_1(x)$ could be found at all. The result shows that a scattering potential is indeed obtained.] To obtain $V_2(x)$ we used the Gelfand-Levitan equation from the right with the reflection coefficient $c(k)$. The functions $\psi(x|k)$ and $\varphi(x|k)$ are the Jost wave functions, respectively, for the left and right Gelfand-Levitan equations.

One of the peculiarities of $V_1(x)$ and $V_2(x)$ is that the reflection coefficients from both sides are equal, though these potentials are not symmetric. Thus, while a sufficient condition for the equality of $b(k)$ and $c(k)$ is that the potential be symmetric, the present example shows that the condition is *not* necessary. One might conjecture that whenever $b(0)=1$, then $b(k)=c(k)$. That this is not true in general is proved by a counterexample to be given in a later paper.

It should be noted that the Gelfand-Levitan equations mentioned in this note are solved by a generalization of the method of Ref. 6.

The lack of uniqueness of the potential with respect to a scattering operator leads to some questions about the uniqueness of the solution of the Korteweg–de Vries equation. Assume that the solution $V(x,t)$ of the Korteweg–de Vries equation at time $t=0$ were $V_1(x)$, say, of our present example. The reflection coefficient $b(k)$ determines the future solution $V(x,t)$ through the use of the Gelfand-Levitan formalism as described in Ref. 7. However, $V_2(x)$ has the same scattering operator and the same reflection coefficient $b(k)$. Which initial is being used by the method of Ref. 7, $V_1(x)$ or $V_2(x)$? Probably the use of the Gelfand-Levitan equation from the left uses $V_1(x)$ as the initial function $V(x,0)$, whereas $V_2(x)$ is the ini-

tial value when the Gelfand-Levitan equation from the right is used.

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Evidence for Large-x Corrections in Quantum Chromodynamics

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We show there is evidence that a method of summing important logarithmic corrections which are significant in the large-x region leads to a superior description of deep-inelastic scattering data (analyzed with use of the evolution equations). Next-to-leading-order calculations can imitate the impact of this summation method, but at high x it appears that there are higher-order and higher-twist corrections which separate those approaches.

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Considerable effort has been devoted to obtain clean and quantitative predictions in quantum chromodynamics (QCD) for various physical processes.[1] Among these, deep-inelastic processes could, in principle, represent an excellent laboratory for such tests. Although QCD predicts an observable logarithmic violation of scaling for the structure functions, there are difficulties which have prevented a completely satisfactory comparison between theory and experiment.[2] Even when considering only the leading-twist contributions in the operator-product expansion (OPE) there are contributions of higher order in α_s, the running coupling constant. Also, the summation[3-9] of certain logarithmic terms (in moments they are characterized as $\alpha_s \ln^2 n$) leads to large corrections in the $x \to 1$ region ($x = Q^2/2p\cdot q$).

We use the evolution equations[10] to analyze the impact of both second-order and $x \to 1$ corrections to the deep-inelastic structure functions, and we consider the relationship between them. We make direct comparisons between theory and experiment for both electron and neutrino deep-inelastic scattering, using Stanford Linear Accelerator Center–Massachusetts Institute of Technology (SLAC-MIT)[11] and CERN–Dortmund-Heidelberg-Saclay (CDHS)[12] collaboration data. We find that there are dramatic indications in the data to support theoretical expectations.

Brodsky and Lepage[3] have observed that large nonleading contributions to the structure functions, which arise in the $x \to 1$ region because of the gluon radiation corrections of the theory,[13] are related to kinematical constraints. Analyzing deep-inelastic scattering, they observed that the use of the correct kinematic constraints plus the correct argument of α_s in evolution equations introduces an additional term. For the moments of the structure functions, this reproduces the sum of the large $\alpha_s \ln^2 n$ corrections (which come from the $x \to 1$ region). These corrections are the same ones as computed with use of the operator-product expansion[8,9] to order α_s. Similar arguments have been developed by Parisi[4] and by

