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Precise SU(5) Predictions for $\sin^2\theta_W^{\exp}$, m_W, and m_Z

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Radiative corrections to ν-induced neutral-current scattering are computed for an arbitrary $G \supset [SU(3)]_c \otimes [SU(2)]_L \otimes U(1)$. Application to the Georgi-Glashow SU(5) model with a single superheavy mass leads, for $\Lambda_{\overline{MS}}$ (the mass scale of quantum chromodynamics) $=0.4$ GeV, to the precise predictions $\sin^2\theta_W^{(\nu_\mu;l)}(0)=0.2104$ for ν_μ-lepton scattering at $q^2=0$ and $\sin^2\theta_W^{(\nu;h)}(-20\text{ GeV}^2)=0.2098$ for deep-inelastic ν-hadron scattering. This result, together with our previous radiative corrections to $m_W \sin\theta_W$, gives the SU(5) predictions $m_W = 84.36$ GeV and $m_Z = 94.91$ GeV.

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A very appealing consequence of grand unification is that $\sin^2\theta_W^0 \equiv e_0^2/(g_2^2)_0$ {the ratio of bare electromagnetic and weak $[SU(2)]_L$ couplings} is elevated from an infinite adjustable counterterm to a definite rational number (in many cases $\sin^2\theta_W^0 = 3/8$).[1] However, the effective value of $\sin^2\theta_W$ measured in present-day experiments can differ significantly from its bare asymptotic value, $\sin^2\theta_W^0$, because of large radiative corrections (finite renormalization effects), a feature pointed out by Georgi, Quinn, and Weinberg.[2] In this Letter we present a general expression for the renormalized quantity $\sin^2\theta_W^{\exp}(q^2)$, incorporating $O(\alpha)$ corrections, which is valid in any grand unified theory (GUT). A byproduct of our analysis is a precise prediction for m_W and m_Z, the masses of the $W^\pm$ and Z^0 intermediate vector bosons.

We adopt a phenomenological definition of the renormalized weak mixing angle in which one merely replaces $\sin^2\theta_W^0$ by $\sin^2\theta_W^{\exp}(q^2)$ and $(g_2^2)_0/8(m_W^0)^2$ by $\rho_{nc}G_\mu/\sqrt{2}$ in weak neutral-current tree amplitudes. [$G_\mu = (1.166\,32 \pm 0.000\,02)\times 10^{-5}$ GeV^{-2} is the muon decay constant.[3]] It has been shown that, aside from QED corrections and certain small induced contributions to ν-hadron scattering which are best treated separately, all corrections of $O(\alpha)$ to neutrino-neutral-current interactions can be incorporated into ρ_{nc} and $\sin^2\theta_W^{\exp}(q^2)$.[4] The superscript exp on $\sin^2\theta_W^{\exp}(q^2)$ signifies that it depends (slightly) on the particular experiment considered and that it is the quantity determined by phenomenological analyses of present-day neutral-current experiments. This function can be related to the momentum- and process-independent renormalized weak mixing angle

$$\sin^2\theta_W = 1 - m_W^2/m_Z^2 \tag{1}$$

used in Refs. 3 and 4 via the relationship

$$\sin^2\theta_W^{\exp}(q^2) = \kappa(q^2)\sin^2\theta_W. \tag{2}$$

The quantity $\kappa(q^2)=1+O(\alpha)$ was previously calculated[4] in the $[SU(2)_L\otimes U(1)$ theory and found to be very near unity and rather insensitive to q^2. Those results remain valid for $G\supset[SU(2)]_L\otimes U(1)$, a fact which may be regarded as an illustration of the decoupling theorem.[5]

We calculate $\sin^2\theta_W{}^{\exp}(q^2)$ as follows: First, a renormalization-group analysis is employed to sum up all the large logarithmic corrections to $\sin^2\theta_W{}^0$. The running couplings, defined by modified minimal subtraction ($\overline{\text{MS}}$), are integrated from the heaviest mass scale in the theory where they are equal down to m_W. This is carried out with use of different effective β functions for the distinct couplings with the stipulation that massive particles decouple from the theory (they no longer contribute to the β functions) as μ, the running mass scale, becomes less than their mass. In that way we obtain an effective low-energy mixing angle $\sin^2\hat\theta_W(m_W)=\hat e^2(m_W)/\hat g_2{}^2(m_W)$ of the form[6-8]

$$\sin^2\hat\theta_W(m_W)=\sin^2\theta_W{}^0[1-\cot^2\theta_W{}^0\hat\alpha(m_W)\sum_i C_i\ln(m_i/m_W)+\cdots]+[\hat\alpha(m_W)/6\pi]\cos^2\theta_W{}^0, \tag{3}$$

where the sum is over all heavy particles ($m_i>m_Z$) in G and the C_i are constants determined from the first terms in the effective β functions of $\hat e$ and $\hat g_2$.[2,6,7] The ellipses in Eq. (3) represent the contribution of higher-order nonleading logarithmic corrections, which can be evaluated by including higher-order terms in the β functions.[7] The effective electromagnetic coupling $\hat\alpha(m_W)=\hat e^2(m_W)/4\pi$ is related to the fine-structure constant $\alpha=1/137.036$ by

$$\hat\alpha^{-1}(m_W)=\alpha^{-1}-(2/3\pi)\sum_f Q_f{}^2\ln(m_W/m_f)+1/6\pi, \tag{4}$$

where the summation is over all fermions, leptons, and quarks with $m_f<m_W$ (a color factor of 3 must be included for quarks) and Q_f is the fermion's electric charge. Next, using $\sin^2\hat\theta_W(m_W)$ in the effective $[SU(2)]_L\otimes U(1)$ theory with heavy particles ($m_i>m_Z$) decoupled, we calculate the remaining $O(\alpha)$ corrections to neutrino–neutral-current scattering[4] employing modified minimal subtraction to eliminate the divergences encountered while setting μ, the dimensional regularization unit of mass, equal to m_W. In that way we find for ν_l ($l=e,\mu,\tau$) neutral-current scattering at $-q^2\ll m_W{}^2$:

$$\sin^2\theta_W{}^{\exp}(q^2)=\sin^2\hat\theta_W(m_W)-\frac{\hat\alpha(m_W)}{2\pi}B-\frac{\alpha(-q^2)}{2\pi}\left[\frac{c^2}{3}+\frac{1}{2}-2J_l(q^2)+\sum_f(C_{3f}Q_f-4s^2Q_f{}^2)J_f(q^2)\right], \tag{5}$$

where $c^2\equiv\cos^2\hat\theta_W(m_W)$, $s^2\equiv\sin^2\hat\theta_W(m_W)$, $J_f(q^2)\equiv\int_0^1dx\,x(1-x)\ln\{[m_f{}^2-q^2x(1-x)]/m_W{}^2\}$ (m_f is the mass of fermion f), C_{3f} is twice the third component of weak isospin (e.g., $C_{3e}=-1$, $Q_e=-1$), and

$$\alpha^{-1}(-q^2)=\alpha^{-1}+\Pi_{\gamma\gamma}(q^2)-\Pi_{\gamma\gamma}(0), \tag{6a}$$

$$\alpha^{-1}(-q^2)\simeq\alpha^{-1}-(2/\pi)\sum_f Q_f{}^2\int_0^1dx\,x(1-x)\ln\{[m_f{}^2-q^2x(1-x)]/m_f{}^2\}. \tag{6b}$$

In Eq. (5) the contributions denoted by B arise from box diagrams and are given by $B=(\frac{5}{2}-61s^2/20-9s^4/10+14s^6/9)/2c^4$ for ν-hadron scattering and $B=(\frac{19}{8}-17s^2/4+3s^4)/c^2$ for ν-lepton scattering.[4]

In obtaining Eq. (5) we included higher-order vacuum polarization $\Pi_{\gamma\gamma}(q^2)$ corrections to the photon propagator, which replaced $\hat\alpha(m_W)$ with $\alpha(-q^2)$ in some of the terms. The hadronic contribution to $\alpha(-q^2)$ in Eq. (6a) can be obtained directly from experimental measurements of $\sigma(e^+e^-\to\text{hadrons})$ via a dispersion relation. Comparison with the analysis of Sirlin[3] shows that to produce an equivalent result from the "free"-quark calculation of Eq. (6b), effective light-quark masses of about $m_u=m_d=m_s=0.1$ GeV should be employed. Using those mass values along with $m_c=m_b/3=1.5$ GeV, $m_t=18$ GeV, and $m_W=84.4$ GeV (which is self-consistent with our final result), we find from Eq. (4) that $\hat\alpha^{-1}(m_W)=127.66$.[9]

Our result in Eq. (5) is applicable to any GUT. We now focus on the simplest SU(5) model[1] with three generations of fermions and all superheavy bosons (vectors and scalars) degenerate with mass m_S. In that case $\sin^2\hat\theta_W(m_W)$ is given by[6,7,10]

$$\sin^2\hat\theta_W(m_W)=\tfrac{3}{8}\left[1-\frac{109}{9}\frac{\hat\alpha(m_W)}{2\pi}\ln\left(\frac{m_S}{m_W}\right)+\cdots\right]+\frac{5\hat\alpha(m_W)}{48\pi}, \tag{7}$$

where the higher-order contributions (indicated by ellipses) have been estimated to increase $\sin^2\hat\theta_W(m_W)$ by about 0.22%.[7] From Eqs. (5) and (7) we obtain the SU(5) prediction for $\sin^2\theta_W{}^{\exp}(q^2)$.

By an algebraic rearrangement valid to $O(\alpha)$ one can rewrite that expression as

$$\sin^2\theta_W^{\exp}(q^2) = \tfrac{3}{8}\left[1 - \frac{109}{9}\frac{\alpha(-q^2)}{2\pi}\ln\left(\frac{m_S}{m_W}\right) + \cdots\right] - \frac{\hat{\alpha}(m_W)}{2\pi}B - \frac{\alpha(-q^2)}{2\pi}\left[\tfrac{1}{2} - 2J_l(q^2) + \sum_f (C_{3f}Q_f - \tfrac{3}{2}Q_f^2)J_f(q^2)\right]. \quad (8)$$

This alternative form has a simple physical interpretation, as it indicates that the higher-order q^2-dependent terms proportional to the large factor $\frac{109}{9}\ln(m_S/m_W)$ can be identified with corrections to the photon propagator and absorbed into $\alpha(-q^2)$. It also illustrates the important fact that $\sin^2\theta_W^{\exp}(q^2)$, when expressed in terms of $\alpha(-q^2)$, is independent of the u-, c-, and t-quark masses since $C_{3f}Q_f - \frac{3}{2}Q_f^2 = 0$ for those flavors. Note, also, that both Eqs. (5) and (8) are independent of the Higgs mass m_φ and that for large $|q^2|$ the hadronic contribution to $\alpha(-q^2)$ can be directly obtained from e^+e^- annihilation data. Thus, aside from the determination of $\ln(m_S/m_W)$, we expect the SU(5) prediction for $\sin^2\theta_W^{\exp}(q^2)$ to have a very small uncertainty.

The superheavy mass m_S of the SU(5) model has been estimated by comparing effective quantum chromodynamic (QCD) and QED couplings. That analysis gives,[7,11] for $\hat{\alpha}^{-1}(m_W) = 127.66$,

$$m_S/m_W = 6.5\times10^{12}[\Lambda_{\overline{\rm MS}}/(0.4\ \text{GeV})]^{1.07}, \quad (9)$$

where $\Lambda_{\overline{\rm MS}}$ is the conventional QCD mass scale obtained with use of modified minimal subtraction[12] with two terms in the effective four-flavor β function. To obtain $\sin^2\theta_W^{\exp}(q^2)$ it is simplest to first set $q^2 = 0$ in Eq. (8). Using $m_W = 84.4$ GeV and including the 0.22% enhancement of $\sin^2\hat{\theta}_W(m_W)$ mentioned after Eq. (7), we obtain, for ν_μ-lepton scattering,

$$\sin^2\theta_W^{(\nu_\mu;l)}(0) = 0.2104 + 0.006\ln[(0.4\ \text{GeV})/\Lambda_{\overline{\rm MS}}]. \quad (10)$$

The corresponding value for ν_μ-hadron scattering is 0.0007 larger. In the case of ν_e scattering at $q^2 = 0$, $\sin^2\theta_W^{\exp}(0)$ is smaller by 1.8%.[4] Present determinations of $\Lambda_{\overline{\rm MS}}$ from deep-inelastic scattering find $\Lambda_{\overline{\rm MS}} = 0.4$ GeV with about a factor of 2 uncertainty which implies an uncertainty of about 0.0042 in our predictions for $\sin^2\theta_W^{\exp}(q^2)$. The q^2 variation of $\sin^2\theta_W^{\exp}(q^2)$ can be obtained from Eqs. (5) and (6) or by using $\sin^2\theta_W^{\exp}(q^2) = [\kappa(q^2)/\kappa(0)]\sin^2\theta_W^{\exp}(0)$, where $\kappa(q^2)$ is the function in Eq. (2) which we studied previously.[4] For $s^2 \simeq 0.21$ and $m_u = m_d = m_s = 0.1$ GeV, we find that $\sin^2\theta_W^{\exp}(q^2)$ montonically decreases by $\simeq 0.9\%$ over the interval $0 \leq -q^2 \leq 100\ \text{GeV}^2$ for ν_μ scattering. [For ν_e scattering, $\sin^2\theta_W^{\exp}(q^2)$ approaches the ν_μ result for $-q^2 \gg m_\mu^2$.] At $q^2 = -20\ \text{GeV}^2$, a good approximation for the average value of q^2 in present-day deep-inelastic ν-hadron scattering experiments,[13] we find

$$\sin^2\theta_W^{(\nu;h)}(-20\ \text{GeV}^2) = 0.2098 \pm 0.0042. \quad (11)$$

The results in Eqs. (10) and (11) provide stringent tests of the SU(5) model. Alternatively, when $\sin^2\theta_W^{\exp}(q^2)$ is precisely determined experimentally, Eq. (8) can be used to pinpoint the value of m_S, thereby reducing the uncertainty in the SU(5) prediction for the proton lifetime ($\tau_p \propto m_S^4$).[6,7] The predictions of other GUT's will in general differ only in $\sin^2\hat{\theta}_W(m_W)$. It is sufficient to insert the appropriate values for $\sin^2\hat{\theta}_W(m_W)$ in Eq. (5) to obtain their predictions. In particular, we note that the results in Eqs. (10) and (11) hold for any GUT with $\sin^2\theta_W^0 = \frac{3}{8}$ which has a single superheavy mass.[14]

How does our prediction for $\sin^2\theta_W^{\exp}(q^2)$ compare with experiment? At present, the best data comes from deep-inelastic ν_μ-hadron scattering. Two-parameter fits yield[13] $\sin^2\theta_W^{(\nu_\mu;h)}(q^2) = 0.232 \pm 0.027$ and $\rho = 0.999 \pm 0.025$, consistent with Eq. (11) within the rather large experimental errors. A more precise determination of $\sin^2\theta_W^{(\nu_\mu;h)}(q^2)$ can be obtained by holding ρ fixed and performing a one-parameter fit. However, since experiments measure the ratio of neutral-current to charged-current cross sections and the phenomenological determination of $\sin^2\theta_W^{\exp}(q^2)$ is sensitive to $\rho = \rho_{\rm nc}/\rho_{\rm cc} = 1 + O(\alpha)$, radiative corrections to both processes must first be carried out.

Using Eqs. (2) and (10) along with our previous expression[4] for $\kappa^{(\nu_\mu;l)}(0)$, we find, for $m_t \simeq 18$ GeV and $m_\varphi \simeq m_Z$,[16]

$$\sin^2\theta_W = 1 - m_W^2/m_Z^2 = 0.2100 + 0.006\ln[(0.4\ \text{GeV})/\Lambda_{\overline{\rm MS}}]. \quad (12)$$

It is rather remarkable how close the calculated values of $\sin^2\theta_W$ and $\sin^2\theta_W^{\exp}(q^2)$ turn out to be.

From Eq. (12) and our previously calculated radiative corrections to $m_W\sin\theta_W$ in the $[\text{SU}(2)]_L \otimes \text{U}(1)$ theory,[3,6,17] we find, for $\Lambda_{\overline{\rm MS}} = 0.4$ GeV,

$$m_W = 84.36\ \text{GeV}, \quad m_Z = 94.91\ \text{GeV}. \quad (13)$$

The dependence of $\sin^2\theta_W$, m_W, and m_Z on $\Lambda_{\overline{\rm MS}}$ is illustrated in Table I. We note that the radiative corrections included in these predictions are

TABLE I. SU(5) predictions for $\sin^2\theta_W$, m_W, and m_Z corresponding to a range of 0.2 GeV $\leq \Lambda_{\overline{MS}} \leq$ 0.8 GeV. The values quoted were obtained with use of $m_t = 18$ GeV and $m_\varphi = m_Z$ (Ref. 16).

$\Lambda_{\overline{MS}}$ (GeV)	$\sin^2\theta_W$	m_W (GeV)	m_Z (GeV)
0.2	0.2142	83.53	94.23
0.3	0.2117	84.01	94.62
0.4	0.2100	84.36	94.91
0.5	0.2086	84.63	95.13
0.6	0.2075	84.86	95.32
0.7	0.2066	85.05	95.48
0.8	0.2058	85.22	95.63

about +3.7%, an important sizable effect.

Ongoing neutrino experiments along with the anticipated discovery of $W^\pm$ and Z^0 followed by measurements of their masses should pinpoint the values of $\sin^2\theta_W{}^{\exp}(q^2)$, $\sin^2\theta_W$, m_W, and m_Z. Agreement with the SU(5) predictions which we have given would be a spectacular triumph for grand unification.

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[8]The correction $\hat{\alpha}(m_W)\cos^2\theta_W{}^0/6\pi$ in Eq. (3) is a remnant of one-loop superheavy-vector-boson contributions left over after the modified minimal subtraction is performed. See S. Weinberg, Phys. Lett. 91B, 51 (1980).

[9]This estimate differs somewhat from earlier results (Marciano, Refs. 6 and 7) because smaller light-quark masses have been used. Our prediction for $\sin^2\theta_W{}^{\exp}(q^2)$ is rather insensitive to this change.

[10]If the physical scalars in SU(5) have mass $m_H \neq m_S$, $\sin^2\hat{\theta}_W(m_W)$ changes by $-[\hat{\alpha}(m_W)/12\pi]\ln(m_H/m_S)$. For $0.1 \leq m_H/m_S \leq 10$ this introduces an uncertainty of about ± 0.0005 in $\sin^2\theta_W{}^{\exp}(q^2)$.

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[14]The results in Eqs. (10) and (11) also hold for larger GUT's containing the SU(5) model and are unchanged [to $O(\alpha)$] by the addition of heavy degenerate multiplets of fermions and scalars.

[15]The corrections to $\rho_{\rm nc}$ have been evaluated and found to be very small (See Marciano and Sirlin, Ref. 4). Those affecting $\rho_{\rm cc}$ are expected to be more significant since they must include the term $(\alpha/\pi)\ln(m_Z/Q)$ present in the corrections to $G_V G_\mu$ [see A. Sirlin, Rev. Mod. Phys. 50, 573 (1978)], where Q is a relevant mass scale such as the momentum transfer or some hadronic mass. This effect alone would decrease ρ and the phenomenologically determined value of $\sin^2\theta_W{}^{(\nu_\mu;h)}(q^2)$. However, to determine Q precisely and the possible existence of large nonlogarithmic terms, a more detailed calculation of the radiative corrections to $\rho_{\rm cc}$ is necessary. That analysis is in progress.

[16]Our prediction for m_W is very insensitive to the values of m_t and m_φ.

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