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Quenched Randomness in the Two-Dimensional Ferromagnetic Planar Model

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A model of a disordered planar model is introduced and solved with use of the replica method. The model is designed to mimic the inhomogeneities known to exist in helium and granular superconducting films. The results indicate that at onset, the universal results, predicted on the basis of renormalization-group analysis, hold to lowest order in the parameter measuring the disorder. The disorder has the important effect of increasing the vortex pairs couplings, resulting in an initial *increase* of the critical temperature. Possible consequences of the results to related experiments are also presented.

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The understanding of superfluidity in two-dimensional (2D) systems has been rapidly expanding in the last few years, from the seminal papers by Kosterlitz and Thouless (KT)[1] and Berezinskii,[2] to the specific[3,4] predictions by Nelson and Kosterlitz (NK).[5] There have been several experiments on helium films by Bishop and Reppy,[6] by Rudnick,[6] by Telschow and Hallock,[6] and by Webster, Webster, and Chester[6] that have tested the NK prediction. Also recent experiments in superconducting films have been interpreted in terms of the KT theory.[7] In spite of this success, there is still some controversy on the interpretation of the experimental results. In particular, Dash[8] has argued strongly that the ^{4}He experiments were done on substrates that are far from homogeneous, such that what has actually been measured is not genuine 2D superfluidity. Dash further says that even if the substrate was flat, the films themselves would not be uniform. The same criticism applies to the experiments done in superconducting films: The materials used in these experiments are either granular or amorphous.

In this communication a model of a disordered planar model is introduced and solved with use of the replica method.[9] The model is thought to mimic a granular superconductor or an inhomogeneous He film. The language of the superconductors will be used throughout while having in mind that the same model applies to the helium case.

Granular superconductors consist of metallic grains embedded in an insulating matrix. At high temperatures the grains are essentially decoupled from each other. When lowering the temperature the grains interact via a Josephson coupling that can lead to long-range correlations. This has led several authors to write the Hamiltonian of a granular superconductor at low temperatures as a ferromagnetic planar model[10]

$$H/k_B T = \sum_{\vec{r},\vec{r}'} J_{\vec{r},\vec{r}'} \cos(\theta_{\vec{r}} - \theta_{\vec{r}'}), \tag{1}$$

where k_B is the Boltzmann constant (set to 1 from now on) and T is the temperature. The variable $J_{\vec{r},\vec{r}'}$ is related to the Josephson coupling between grains. Early approximations[10] took $J_{\vec{r},\vec{r}'}$ as a nearest-neighbor (nn) constant. However, in a more realistic model $J_{\vec{r},\vec{r}'}$ is a random variable arising from the nonuniform size and shape distribution of the grains. Here $J_{\vec{r},\vec{r}'}$ will be taken as a nn independent random variable (irv). The justification for taking $J_{\vec{r},\vec{r}'}$ as a nn variable is similar in spirit to that given by Edwards and Anderson to justify their model for a spin-glass.[9] Start by defining $J_{\vec{r},\vec{r}'}$ as

$$J_{\vec{r},\vec{r}'} = J_0 \exp(-z_{\vec{r},\vec{r}'}). \tag{2}$$

Here $z_{\vec{r},\vec{r}'}$ is a positive real irv with normalized probability distribution

$$P(z_{\vec{r},\vec{r}'}) = (2/x\pi)^{1/2}\theta(z_{\vec{r},\vec{r}'})\exp(-z_{\vec{r},\vec{r}'}{}^2/2x), \qquad (3)$$

and θ is the Heaviside step function. J_0 is the maximum value allowed to $J_{\vec{r},\vec{r}'}$. The physical meaning of x is clear: When x is large the grains are weakly coupled, and if x is small the value of $J_{\vec{r},\vec{r}'}$ fluctuates around J_0. x is a parameter that can be varied experimentally, in principle. In this paper the limit when x is small will be considered. It is easy to show that the probability law for $J_{\vec{r},\vec{r}'}$ is of the log-normal form. This type of probability law has been shown to be universal in 1D quantum random problems, for which the scaling variable is $\ln(J_{\vec{r},\vec{r}'}/J_0)$.[11]

Substituting Eq. (1) into Eq. (2) it follows

$$H/T = \sum J_0 \cos(\theta_{\vec{r}} - \theta_{\vec{r}+1} + iz_{\vec{r},\vec{r}+1}). \qquad (4)$$

This Hamiltonian is strictly valid in the limit when x is small. At low temperatures it is convenient to take the Villain approximation of Eq. (4), giving[4]

$$H_v/T = -\tfrac{1}{2}\sum J_0(\theta_{\vec{r}} - \theta_{\vec{r}+1} + 2\pi m_{\vec{r},\vec{r}+1} + iz_{\vec{r},\vec{r}+1})^2, \qquad (5)$$

with $i=\sqrt{-1}$ and $m_{\vec{r},\vec{r}+1}$ is the integer dynamic link variable for the vortex excitations. The difference between this Hamiltonian and the pure case Hamiltonian is the addition of an *imaginary continuous* independent random gauge variable. Equations (3) and (5) define the model to be studied in this paper. The Hamiltonian is a good approximation up to T_{KT}. As usual we are interested in taking the quenched average with respect to Eq. (3). The importance in writing the model in this way is that the randomness has been removed from the coupling to a link variable. From the technical point of view, it means that when taking the average with respect to $P(z_{\vec{r},\vec{r}+1})$ a Gaussian Hamiltonian will be mapped into a Gaussian Hamiltonian with renormalized coupling constants. This is in contrast to the usual replica-method calculations that, after configurational averaging, generate higher-order terms. This simplification allows to solve the model explicitly.

The quenched free energy is

$$(F/T)_z = -\int d\{P(z)\}\ln\int d\{\theta,m\}e^{H_v(\theta,m,z)/T}, \qquad (6)$$

where $\theta\in[0,2\pi]$ and $m=0,\pm1,\pm2,\ldots$. The configurational average in Eq. (6) is performed with use of the replica method,[9]

$$(F/T)_z = -\lim_{n\to 0}[(1/n)\int d\{z\}\theta\{z\}\int d\{U\}e^{\tilde{H}(\vec{U},z)/T}]. \qquad (7)$$

To simplify the notation, the n-vector $\vec{U}=(U^1, U^2,\ldots,U^n)$ has been defined with components $U_{\vec{r}}{}^\alpha = (\theta_{\vec{r}}{}^\alpha - \theta_{\vec{r}+1}{}^\alpha + 2\pi m_{\vec{r},\vec{r}+1}{}^\alpha)$, and the meaning of the measure $d\{\vec{U}\}$ follows from the one given above. The effective Hamiltonian $\tilde{H}(\vec{U},z)$ reads

$$\tilde{H}(\vec{U},z) = -\tfrac{1}{2}J_0\sum_{\alpha=1}^{n}[U^\alpha+2iz, U^\alpha] - (1/2x')[z,z]. \qquad (8)$$

The brackets stand for the scalar product defined as $[a,b]\equiv\sum a_{\vec{r},\vec{r}+1}b_{\vec{r},\vec{r}+1}$. The variable x' is equal to

$$x' = x/(1-nJ_0x). \qquad (9)$$

It is clear from Eq. (9) that Eq. (8) will lead to sensible results only for positive values of x'. This means for values of the temperature $J_0^{-1} > nx$. If $J_0^{-1}\gg nx$, then $x'\to x$ and, when $J_0^{-1}\sim x^+n$, it corresponds to the limit $x'\to\infty$. The integral over z is computed, taking into account the θ-function constraint, and after some algebra, $(F)_z$ becomes

$$(F/T)_z = -\lim_{n\to 0} n^{-1}\int d\{\vec{U}\}\exp(-\tfrac{1}{2}J_0\sum_\alpha\sum_\beta[U^\alpha R_{\alpha\beta}, U^\beta]), \qquad (10)$$

with the $n\times n$ matrix

$$R_{\alpha\beta} = \delta_{\alpha\beta} + c(x')J_0x'S_{\alpha\beta}. \qquad (11)$$

$\delta_{\alpha\beta}$ is the unit matrix and $S_{\alpha\beta}$ has all its entries equal to 1. Equation (10) was obtained in the limits when $x'\to\infty$ and $x'\to x$, the only difference being the constant $c(x'\to\infty)=1$ and $c(x'\to x)=\frac{1}{2}$. $c=1$ will be taken from now on, without changing the qualitative nature of the results. Also, logarithmic corrections to the Hamiltonian in Eq. (10) have been neglected. The orthogonal transformation $\tilde{U}^\alpha = \sum_\beta T_\beta{}^\alpha U^\beta$, such that $\sum_{\lambda,\epsilon}T_\alpha{}^\lambda R_{\lambda\epsilon}T_\beta{}^\epsilon = \lambda_{(\alpha)}\delta_{\alpha\beta}$, reduces Eq. (10) to

$$(F/T)_z = -\lim_{n\to 0} n^{-1}\int d\{\tilde{U}\}\exp(-\tfrac{1}{2}J_0\sum_\alpha\lambda_{(\alpha)}[\tilde{U}^\alpha,\tilde{U}^\alpha]). \qquad (12)$$

The Jacobian of the orthogonal transformation leaves the volume element invariant. It is easy to diagonalize Eq. (11). There are two types of eigenvalues. $\lambda_1 = 1 + nJ_0x'$ and $\lambda_2 = 1$. The first has degeneracy of 1; and the second, of $n-1$. This result leads immediately to

$$(F/T)_z = -\lim_{n\to 0} n^{-1}(\int d\{U^1\}\exp(-\tfrac{1}{2}\tilde{J}_0[\tilde{U}^1,\tilde{U}^1]))^n \quad (13)$$

in which the renormalized coupling

$$\tilde{J}_0 = J_0[1 + J_0x/(1-nxJ_0)]. \quad (14)$$

Equation (13) indicates that the random problem has been mapped into a nonrandom problem, with the renormalized coupling constant $\tilde{J}_0$ dependent on the disorder parameter x. Note that when $x \to 0$ the model reduces to the pure case, as it should. This somewhat unexpected mapping can be traced back to the way in which disorder was introduced in the model. Moreover, it follows from Eq. (14) that disorder has the effect of producing a stronger coupling when the temperature is lowered; more discussion on the physics of this fact will be given at the end of this communication.

Borrowing from the well-known results for the periodic case,[3,4] the recursion relations for this problem can be written at once in the limit when $n\to 0$. The renormalized vortex pair density is given by

$$y(x) \sim \exp[-(\pi^2/2)J_0(1+J_0x)]. \quad (15)$$

It follows from the recursion relations that the critical temperature as a function of x is[12]

$$J_{0\mathrm{KT}}^{-1} = \tfrac{1}{2}\pi + x - \tfrac{1}{2}x^2 + O(x^3). \quad (16)$$

The *increment* in the critical temperature with disorder x, for small x, can be understood in the following way: x gives a measure of the width of the probability for $J_{\vec{r},\vec{r}+1}$. When x increases, $\{J_{\vec{r},\vec{r}+1}\}$ become, on the average, weaker but the effective $\tilde{J}_0(x)$ increases. $\tilde{J}_0$ measures the strength of the interaction between renormalized vortex pairs, and the temperature for the unbinding of the vortices has to be higher. The decrease in the vortex density with disorder is due to the potential barrier for the nucleation of the vortices because of the anisotropy field created by the random J's.

An immediate consequence from Eq. (13) is the invariance of the Nelson-Kosterlitz universality prediction when introducing disorder. NK pointed out that the ratio $\rho_n(T_{\mathrm{KT}})/T_{\mathrm{KT}}$ is equal to a universal constant, with ρ the superfluid density. Here it is clear that, although T_{KT} changes, the ratio $\rho(T_{\mathrm{KT}}(x))/T_{\mathrm{KT}}(x)$ is equal to the same constant. Note that this statement is strongly dependent on having mapped the problem onto an equivalent planar model that leads to the same type of scale-invariant recursion relations. As a by-product, the value of the critical exponent $\eta(T_{\mathrm{KT}}(x))$ remains equal to $\frac{1}{4}$.

The above discussion has centered around T_{KT} where the effects of disorder are quantitative but not qualitative. However, when lowering the temperature it is seen from Eq. (15) that the vortex density is reduced by the presence of disorder. Although some of these results are in agreement with the Harris criteria,[13] the special nature of the phase transition in the planar model requires an explicit treatment of the effects of disorder in this model. In particular, the initial increase of T_{KT} with x is unusual.

The KT theory when applied to superconductors consists of two separate parts.[7a] On the one hand it is the determination of T_{KT} and the proof of the universality relation for $\rho(T_{\mathrm{KT}})/T_{\mathrm{KT}}$, and on the other the connection to the superconducting transition temperature T_{BCS}, which is a parameter to be fed into the theory. Experimentally what is measured is the ratio $T_{\mathrm{KT}}/T_{\mathrm{BCS}}$ as a function of sheet resistance per square, $R_\square$. In the experimental results reported by Beasley, Mooij, and Orlando[7a] and Hebard and Vanderberg, the decay of $T_{\mathrm{KT}}/T_{\mathrm{BCS}}$ as a function of $R_\square$ is much slower than the one predicted by the equation of Beasley, Mooij, and Orlando. Here we notice that $[T_{\mathrm{KT}}(x)/T_{\mathrm{BCS}}] > [T_{\mathrm{KT}}(0)/T_{\mathrm{BCS}}]$, which would seem to be in qualitative agreement with having a slower decay. Note, however, that the region of validity of the calculation presented in this paper corresponds to having small $R_\square$ for which detailed experiments are yet to be done. Also, in the limit when $R > 13$ kΩ, quantum effects may come into play.[14]

In summary, a simple model of a random planar model has been introduced and solved for small disorder. The model is believed to represent the situation encountered in inhomogeneous helium films and granular superconductors. The solution yields further support to the interpretation of recent experiments[6,7] in terms of the Kosterlitz-Thouless mechanism. Away from T_{KT}, however, it is found that having disorder limits the spontaneous nucleation of vortex pairs, as well as making their coupling stronger. The limits of applicability of the approximate model considered here are not well understood at present. However, it is possible to argue that the effects of disorder may in fact have helped in the success of the experiments by reducing the deviations from the Gaussian approximation inherent in the KT theory.

The analysis presented here is purely static and may be dependent on the particular way that disorder was introduced, but the universality of the results should persist. It is possible that some of the discrepancies that still remain between theory and experiment are due to disorder. The details of the calculations in this paper as well as other studies of disorder in the planar model can be found elsewhere.[12,15]

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