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Thermodynamics of an Ultrarelativistic Ideal Bose Gas

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We discuss the properties of an ideal relativistic Bose gas with nonzero chemical potential μ (i.e., with net charge) but differ from all previous discussions by including the effects of antiparticles. We obtain for the first time the relevant high-temperature expansion, emphasizing the constraint $|\mu| \leq m$, and show that a box of massless particles can have a net charge even though $\mu = 0$. Finally, we discuss the properties of Bose-Einstein condensation for both massless and massive bosons in d space dimensions.

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There have recently been a number of papers[1-3] which discuss the properties of an ideal relativistic Bose gas with nonzero chemical potential μ. Particular attention has been focused on the behavior of the Bose-Einstein condensation and the nature of the phase transition in d space dimensions. The ground work for these recent papers was laid many years ago in the work of Juttner[4] and Glaser[5] and more recently by Landsberg and Dunning-Davies[6] and Nieto.[7] These past works have all been in the context of relativistic quantum mechanics. At temperatures larger than the mass of the particles, quantum field theory requires the inclusion of particle-antiparticle pair production.

To describe an ideal Bose gas in the grand canonical ensemble, the natural expression for the number of bosons N in relativistic quantum mechanics is

$$N = \sum_{\vec{k}} \frac{1}{\exp[\beta(E_k - \mu)] - 1}, \tag{1}$$

where $E_k = (k^2 + m^2)^{1/2}$ and $\beta = T^{-1}$ (in units of $\hbar = c = k = 1$), and we must require $\mu \leq m$ in order to ensure a positive-definite value for n_k, the number of bosons with momentum k. The assumption made here is that N is a conserved quantity so that it makes sense to talk of a box of N bosons. This can no longer be true once $T \gtrsim m$; at such temperatures the production of particle-antiparticle pairs becomes important. If $\bar{N}$ is the number of antiparticles, then N and $\bar{N}$ by themselves are not conserved but N-$\bar{N}$ is conserved. Therefore the high-temperature limit of (1) is not relevant in realistic physical systems.[8]

More generally, one may consider any ideal Bose gas with a conserved quantum number (which we will refer to generically as "charge"). The conserved quantum number corresponds to a quantum mechanical operator $\hat{Q}$ which commutes with the Hamiltonian $\hat{H}$. All thermodynamic quantities may be obtained from the grand partition function Tr $\{\exp[-\beta(\hat{H} - \hat{Q})]\}$ considered as a function of T, V, and μ.[9] *Ab initio*, one might think that particles and antiparticles could have independent chemical potentials. However, the fundamental structure of relativistic field theories requires that if the eigenvalue of $\hat{Q}$ is +1 for particles, it must be −1 for antiparticles (i.e., all additive quantum numbers are reversed).

The formula for the net conserved charge, which replaces (1) in quantum field theory, is

$$Q=\sum_{\vec{k}}\left[\frac{1}{\exp[\beta(E_k-\mu)]-1}-\frac{1}{\exp[\beta(E_k+\mu)]-1}\right]. \tag{2}$$

Note that if we regard the expression above as a sum over $n_k-\bar{n}_k$, then we may loosely say that particles and antiparticles have chemical potentials which are opposite in sign. More precisely, only one chemical potential μ describes a system of bosons—the sign of μ indicates whether particles outnumber antiparticles or vice versa. Equally important is the realization that both n_k and $\bar{n}_k$ must be positive definite. This leads to the important conclusion that $|\mu|\leq m$.[10,11]

Before turning to a general analysis, let us quickly use (2) to learn something about Bose-Einstein condensation of the relativistic Bose gas. As in the usual analysis,[9] the sum over k in (2) is converted to an integral so that the charge density $\rho\equiv Q/V$ becomes

$$\rho=\frac{1}{2\pi^2}\int_0^\infty k^2dk\left[\frac{1}{\exp[\beta(E_k-\mu)]-1}-(\mu\to-\mu)\right]. \tag{3}$$

Note that (3) is really an implicit formula for μ as a function of ρ and T. For T above some critical temperature T_c, one can always find a μ (satisfying $|\mu|<m$) such that (3) holds. Below T_c, no such μ can be found and we interpret (3) as the charge density of the excited states: $\rho-\rho_0$, where ρ_0 is the charge density of the ground state. The critical temperature T_c at which Bose-Einstein condensation occurs corresponds to $\mu=\pm m$ (the sign depending on the sign of ρ). Thus, we set $|\mu|=m$ in (3) and obtain an implicit equation for T_c in terms of ρ. In the region $T_c\gg m$ we easily obtain

$$|\rho|\approx\frac{m}{\pi^2T}\int_0^\infty k^2dk\frac{e^{\beta k}}{(e^{\beta k}-1)^2}=\tfrac{1}{3}mT^2, \tag{4}$$

which implies

$$T_c=(3|\rho|/m)^{1/2}. \tag{5}$$

Below T_c, (4) is an equation for $\rho-\rho_0$, so that the charge density in the ground state is

$$\rho_0=\rho[1-(T/T_c)^2]. \tag{6}$$

We note that (5) leads to the important result that any ideal Bose gas of mass m will Bose-Einstein condense at a relativistic temperature (i.e., $T_c\gg m$), provided that $\rho\gg m^3$. Conversely, in the nonrelativistic regime we may apply the standard textbook results[9] to see that $T_c\ll m$, provided that $\rho\ll m^3$.

The T^2 behavior in (6) is to be compared with a T^3 behavior one finds in the literature.[1-6] The difference is due to the fact that previous authors have based their analyses on (1) rather than (2). It is important to notice that if T_c is calculated from (1) it will be independent of m, in contrast to our result (5). This observation has important consequences for the $m\to0$ limit. Consider the following observation: If $\mu=0$, then (2) requires that $Q=0$. But what about the case of charged massless bosons[12]: The requirement $|\mu|\leq m$ implies that $\mu=0$ for massless particles; so how can one have a net charge? The answer is found in (5) and (6): For $m=0$, it follows that $T_c=\infty$ and hence $\rho=\rho_0$. That is, all net charge of an ideal gas of massless bosons resides in the Bose-Einstein–condensed ground state. (Of course, for a photon gas no conserved quantum number exists so that Bose-Einstein condensation does not take place.)

To discuss more fully the relativistic ideal Bose gas requires calculating the high-temperature expansion of (3) and of other thermodynamic variables. For the case $\mu=0$, the high-temperature limit was computed by Dolan and Jackiw.[13] The case of $\mu\neq0$ is far more complicated; Arago de Carvalho and Rosa[2] discuss errors in the previous attempts to calculate the high-temperature limit of (1). They obtain an expansion of (1) which they note is valid only for $\mu<0$; hence their expansion is not applicable to (2).

The first step is to introduce dimensionless variables $x\equiv\beta k$, $\bar{m}\equiv\beta m$, and $r=\mu/m$ (note that $|r|\leq1$). The integrals that must be evaluated are of two types:

$$G_l(\bar{m},r)=\frac{1}{\Gamma(l)}\int_0^\infty x^{l-1}dx\left[\frac{1}{\exp[(x^2+\bar{m}^2)^{1/2}-r\bar{m}]-1}-(r\to-r)\right],$$
$$H_l(\bar{m},r)=\frac{1}{\Gamma(l)}\int_0^\infty\frac{x^{l-1}dx}{(x^2+\bar{m}^2)^{1/2}}\left[\frac{1}{\exp[(x^2+\bar{m}^2)^{1/2}-r\bar{m}]-1}+(r\to-r)\right]. \tag{7}$$

For example, the pressure is $P=(4T^4/\pi^2)H_5$ and the charge density is $\rho=(T^3/\pi^2)G_3$. The functions G_l and H_l satisfy recursion relations:

$$dG_{l+1}/d\bar{m}=lrH_{l+1}-\bar{m}l^{-1}G_{l-1}+\bar{m}^2rl^{-1}H_{l-1},$$
$$dH_{l+1}/d\bar{m}=rl^{-1}G_{l-1}-\bar{m}l^{-1}H_{l-1}. \tag{8}$$

The high-temperature expansion we seek is an expansion in $\bar{m}\ll 1$ (for any $|r|\leq 1$). If one simply expands the integrands of (7) in powers of $\bar{m}$, the result is a power series whose coefficients are, in general, divergent integrals. To avoid this difficulty, we make use of the following identity[13]:

$$\frac{1}{e^y-1}=\frac{1}{y}-\frac{1}{2}+2\sum_{k=1}^{\infty}\frac{y}{y^2+(2\pi k)^2}. \tag{9}$$

A convergence factor $x^{-\epsilon}$ is inserted to ensure that term by term, all integrations are finite. At the end of the calculation we may safely take $\epsilon\to 0$. Applying (9) to the calculation of G_1 and H_1 yields the following expansions in $\bar{m}$:

$$G_1(\bar{m},r)=\frac{\pi r}{(1-r^2)^{1/2}}-r\bar{m}+2\pi r\sum_{k=1}^{\infty}a_k\zeta(2k+1)\left(\frac{\bar{m}}{2\pi}\right)^{2k+1}(-1)^{k+1}, \tag{10}$$

$$H_1(\bar{m},r)=\frac{\pi}{\bar{m}(1-r^2)^{1/2}}+\ln\left(\frac{\bar{m}}{4\pi}\right)+\gamma\sum_{k=1}^{\infty}b_k\zeta(2k+1)\left(\frac{\bar{m}}{2\pi}\right)^{2k}(-1)^k, \tag{11}$$

where $\bar{m}$ has been taken positive. In the above, γ is Euler's constant; $a_1=1$, $a_2=2r^2+\frac{3}{2}$, $b_1=r^2+\frac{1}{2}$, $b_2=r^4+3r^2+\frac{3}{8}$, etc.; and $\zeta(2k+1)$ is Riemann's zeta function. Knowledge of G_1 and H_1 along with the recursion relations (8) is sufficient to obtain all the necessary functions which describe the thermodynamics in three space dimensions.[14] The results[15] (for $T>T_c$) are

$$P=\frac{\pi^2T^4}{45}-\frac{T^2(m^2-2\mu^2)}{12}+\frac{T^2(m^2-\mu^2)^{3/2}}{6\pi}+\frac{\mu^2(3m^2-\mu^2)}{24\pi^2}+\frac{m^4}{16\pi^2}\left[\ln\left(\frac{m}{4\pi T}\right)+\gamma-\frac{3}{4}\right]+O\left(\frac{m^6}{T^2},\frac{m^4\mu^2}{T^2}\right), \tag{12}$$

$$\rho=\frac{1}{3}\mu T^2-\frac{\mu T(m^2-\mu^2)^{1/2}}{2\pi}+\frac{\mu(3m^2-2\mu^2)}{12\pi^2}+O\left(\frac{\mu m^4}{T^2}\right), \tag{13}$$

$$\frac{S}{V}=\frac{4\pi^2T^3}{45}-\frac{T(m^2-2\mu^2)}{6}+\frac{(m^2-\mu^2)^{3/2}}{6\pi}-\frac{m^4}{16\pi^2T}+O\left(\frac{m^6}{T^3},\frac{m^4\mu^2}{T^3}\right), \tag{14}$$

and the energy is $U=TS-PV+\mu\rho V$. For $m=\mu=0$ we recover the usual photon results.[16] Clearly, too, our earlier discussion in (4)–(6) of the phase transition is verified in (13). From these results we can show that the specific heat, c_V, is continuous at T_c but has a discontinuity in its derivative given by

$$\left.\frac{dc_V}{dT}\right|_{T_c^+}-\left.\frac{dc_V}{dT}\right|_{T_c^-}=\frac{-32\pi^2Q}{9m}. \tag{15}$$

It is also of interest to investigate Bose-Einstein condensation in an arbitrary number of dimensions. The general formula for the charge density in d space dimensions is

$$\rho=\pi^{-(d+1)/2}\Gamma((d+1)/2)T^dG_d(\bar{m},r). \tag{16}$$

Consider first thermodynamics in $d=2$ space dimensions. Note that (10) and (8) only give us G_d for odd d. Separate calculations are needed for G_2 and H_2 in order to obtain [by using (8)] the thermodynamic quantities in the case of even d. The methods are the same as before and we find that[15]

$$\rho=\frac{-\mu T}{2\pi}\ln\left(\frac{m^2-\mu^2}{T^2}\right)+\frac{\mu T}{\pi}+O\left(\frac{\mu m^2}{T}\right). \tag{17}$$

If $m\neq 0$, then for finite ρ the point $|\mu|=m$ can never be reached for any value of T. That is, there is no Bose-Einstein condensation in two dimensions for massive particles (one can see that $T_c=0$ as $\mu\to m\neq 0$). However, because $|\mu|\leq m$, the limit $m\to 0$ (which forces $\mu\to 0$) is not smooth. It is consistent to have $T_c\sim m^{-1+\delta}$ $(0<\delta<1)$ and satisfy (17) in the limit of $\mu\to m\to 0$. This indicates that $T_c=\infty$ for massless particles and hence Bose-Einstein condensation does occur for massless bosons in two dimensions. Alternatively, if we follow the logic of Landau and Wilde[17] in the case of $\mu=m=0$, we would write

$$\rho-\rho_0=\lim_{r\to 1}\lim_{\bar{m}\to 0}\frac{T^2}{2\pi}G_2(\bar{m},r). \tag{18}$$

Using (17) and (18) we conclude that $\rho=\rho_0$; i.e., all net charge is in the Bose-Einstein–condensed ground state. The conclusion that massless particles condense in two dimensions agrees with recent claims in the literature.[18]

Lastly, we consider the case of one space dimension. We may obtain the charge density immediately from (10) since $\rho = TG_1/\pi$. It is clear that there is no Bose-Einstein condensation no matter what the mass is. Note that in this case, massless particles can have net charge density even though $\mu = 0$ and no condensation takes place. The reason is that in this case the parameter $r = \mu/m$ survives the $\mu \to 0$, $m \to 0$ limit and can characterize a nonzero charge density.

In summary we have given for the first time the high-temperature expansion of an ideal relativistic Bose gas when $\mu \neq 0$, taking antiparticles into account. Although the expansions of thermodynamic quantities are not analytic at $m = 0$, they do possess smooth limits as $m \to 0$ (as long as we remember that $|\mu| \leq m$); in this limit, we recover the photon-gas results. Furthermore, these expansions allow us to study Bose-Einstein condensation which takes place when $|\mu| \to m$.

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[8]In realistic physical systems, one must of course allow for interactions. For example, if λ characterizes the interaction strength, then (2) is the leading term in an expansion in λ. The ideal-gas results are applicable in weakly coupled systems ($\lambda \ll 1$); but because no realistic system has λ exactly zero (or else thermal equilibrium could not be attained, it is not useful to decouple the antiparticles as in (1).

[9]E.g., see K. Huang, *Statistical Mechanics* (Wiley, New York, 1963).

[10]In finite-temperature quantum field theory, the grand partition function is expressed in terms of a path integral. [See, e.g., C. Bernard, Phys. Rev. D 9, 3312 (1974).] The requirement $|\mu| \leq m$ is necessary for the path integral to converge.

[11]The μ that appears in this paper is related to the chemical potential of nonrelativistic thermodynamics by $\mu_{NR} = \mu - m$. Thus $|\mu| \leq m$ is equivalent to $-2mc^2 \leq \mu_{NR} \leq 0$ (where we have put back the speed of light). In the nonrelativistic limit ($c \to \infty$) we regain the familiar result $-\infty < \mu_{NR} < 0$. More precisely, the nonrelativistic limit corresponds to $T \ll m$. In this limit the contribution of the second term (i.e., the antiparticles) in (2) is exponentially small and we obtain the standard textbook results.

[12]Although no charged massless bosons are known to exist at present, one has good reason to believe that such bosons did exist in the very early universe [see, e.g., A. Linde, Rep. Prog. Phys. 42, 389 (1979)].

[13]L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).

[14]One needs to use the condition that $H_{l+1}(0,0) = 2\zeta(l)/l$ for $l > 1$.

[15]Details will be presented in a forthcoming publication.

[16]When $\mu = 0$, particles and antiparticles contribute equally. In comparing with the photon gas, the two polarization states compensate the fact that the photon is its own antiparticle.

[17]L. J. Landau and I. F. Wilde, Commun. Math. Phys. 70, 43 (1979).

[18]See Refs. 1 and 3. The basic argument of these authors concerns the fact that $k^{d-1}/(\exp\{\beta[(k^2+m^2)^{1/2} - m]\} - 1)$ satisfies the following property: If $m \neq 0$, then the expression is integrable near $k = 0$ for $d > 2$; whereas for $m = 0$, the requirement is $d > 1$.