Confinement in Quantum Chromodynamics

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A proof of confinement in quantum chromodynamics is presented that relies only upon general properties of the theory, such as asymptotic freedom and analyticity.

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It is the purpose of this paper to demonstrate that continuum quantum chromodynamics (QCD) confines quarks and gluons. Our method employs a *reductio ad absurdum* argument whereby the *presumed* existence of unconfined quarks is shown to be inconsistent with calculable *ultraviolet* properties of the theory. We shall attempt to show that no generalized Lehmann-Symanzik-Zimmermann (LSZ) prescription[1] exists for obtaining physical matrix elements from Green's functions when there are quarks (or gluons) in the external state. Our argument strongly suggests that it is not possible to factor out external quark (or gluon) mass singularities, regardless of their structure, when these particles go "on shell." A crucial ingredient to the proof is the assumption that if quarks *can* in fact go on shell then suitably defined amplitudes satisfy standard dispersion relations with singularities dictated solely by unitarity.[2] Unfortunately, our proof sheds little light on the precise *mechanism* of confinement, which, of course, ultimately one would like to understand. By the very nature of the problem it is clear that one cannot proceed without knowing something about the infrared region. Now, in spite of the fact that QCD is not infrared stable it does possess infrared properties that are known with surety and these are that quarks (or gluons) carry conserved quantum numbers. For example, no matter how bizarre the infrared behavior of the theory the charge of a "near-mass-shell" quark is a known, conserved quantity.

We shall begin, therefore, by *assuming* that QCD is a "normal" theory so that, for example, a quark *can* go on shell in the usual way and then examine what happens if we attempt to probe its structure with a current, j_μ, conserved by virtue of the equations of motion. Consider, then, the following vertex function:

$$G_\mu(p,p')u(p) \equiv \int d^4x\, e^{iq\cdot x}\langle 0|\,T[j_\mu(x)\psi(0)]\,|p\rangle. \quad (1)$$

Here, $|p\rangle$ is an assumed free-quark state of momentum p and mass M; i.e., $p^2=M^2$. We normalize the interpolating quark field ψ such that $\langle 0|\psi|p\rangle=1$. In the usual LSZ formalism,[1] $\Gamma_\mu \equiv (\not{p}'-M)G_\mu$ reduces to the mass-shell vertex function, free of singularities in p'^2 ($\equiv W^2$), when the limit $\not{p}'\to M$ is taken; its most general form is[3]

$$\Gamma_\mu(p,p') = [(\not{p}'-W)/2W][F_1(q^2,W)\gamma_\mu + F_2(q^2,W)i\sigma_{\mu\nu}q^\nu + F_3(q^2,W)q_\mu] + (W\to -W). \quad (2)$$

Note that Γ_μ satisfies the following Ward identity:

$$q^\mu\Gamma_\mu(p,p') = (\not{p}'-M). \quad (3)$$

In terms of the F's, this reads

$$[F_1(q^2,W)-1](W-M) = q^2F_3(q^2,W). \quad (4)$$

Now, the presumed existence of (1) can be combined with causality in the standard way to establish fixed $q^2<0$ analyticity properties for the F_i in the upper and lower halves of the cut complex W plane.[2,3] Furthermore, the assumption of spectral conditions allows one to write dispersion relations for the $F_i(q^2,W)$. Their absorptive parts can be obtained from

$$\mathrm{Im}\Gamma_\mu(p,p')u(p) = \tfrac{1}{2}\sum_N (\not{p}-M)\langle 0|\psi|N\rangle\langle N|j_\mu|p\rangle(2\pi)^4\delta^{(4)}(p_n-p'), \quad (5)$$

where the $|N\rangle$ represent a complete set of states. The Schwarz inequality can be used to bound the $\mathrm{Im}F_i$ in terms of products of the quark spectral functions[1] $\rho_i(W^2)$ and their inelastic structure functions $W_i(q^2,W^2)$. Inserting these bounds in a dispersion relation leads to bounds on the $F_i(q^2,W)$ themselves. Such bounds were already derived some time ago[4]; however, these were outside of the context of a gauge theory, and this presents some special problems. In a Lorentz gauge, tacitly assumed in writing

Eq. (2), for instance, the Hilbert space contains states of negative norm which forbids the use of a Schwarz inequality. Thus, if we are to bound Eq. (5) in this way, a gauge must be chosen which is ghost-free. A convenient set of such gauges is the axial one defined by $n^\mu A_\mu = 0$ where A_μ is the gluon field and n^μ an arbitrary vector of fixed nonzero length.[5] Since Γ_μ is a *gauge-dependent* quantity, it, and consequently the F_i, now have an implicit n dependence. The development can be considerably simplified if n is chosen to be perpendicular to the hyperplane defined by p' and q. The most general form for Γ_μ is [instead of Eq. (2)] then

$$\Gamma_\mu(p,p',n) = \left(\frac{\not{n}+n}{2n}\right)\left[\left(\frac{\not{p}'-W}{2W}\right)\{F_1\gamma_\mu + F_2 i\sigma_{\mu\nu}q^\nu + F_3 q_\mu + F_4 n_\mu\} + (W \to -W)\right] + (n \to -n). \tag{6}$$

It is easy to check that this decomposition leaves Eq. (4) unchanged, except, of course, for the added implicit n dependence. We shall need the (positive definite) quark spectral function:

$$\rho_+(W^2,n) \equiv \frac{1}{\pi W}\,\mathrm{Tr}\left(\frac{\not{p}'+W}{2W}\right)\left(\frac{\not{n}+n}{2n}\right)\mathrm{Im}S_F(p',n). \tag{7}$$

It is actually more convenient to use linear combinations of the F_i analogous to G_E and G_M; for example, define $G_E = F_1 + q^2(W+M)^{-1}F_2$ then

$$|\,\mathrm{Im}G_E(q^2,W,n)|^2 \le \frac{8\pi^2 MWQ^2}{(W-M)^2+Q^2}\left(\frac{W-M}{W+M}\right)^2 \rho_+(W^2,n)W_L(Q^2,W^2). \tag{8}$$

Here $Q^2 \equiv -q^2 \ge 0$ and W_L is the longitudinal quark structure function.

We now exploit the analyticity properties of the theory which together with the Ward identity, Eq. (4), lead to a "sum rule" of the form

$$1 - F_1(Q^2,M) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dW\,\frac{\mathrm{Im}F_1(Q^2,W)}{W-M}. \tag{9}$$

This is valid provided, of course, the integral converges. The normalization, 1, which plays a crucial role in the ensuing argument represents the static quark charge $F_1(0,M)$. Note that because the F_i were defined through Γ_μ rather than G_μ the integration over W excludes the contribution from the single particle singularity at $W=M$ (here *assumed* to be a simple pole). A similar equation can be derived for G_E except that there is an added contribution on the left-hand side arising from the kinematic singularity at $W=-M^4$; indeed, $F_1(Q^2,M)$ in (9) is replaced by $G(Q^2) \equiv G_E(Q^2,M) - (Q^2/2M)F_2(Q^2,-M)$. Equations (8) and (9) can now be combined to derive the following inequalities:

$$|1 - G(Q^2)| \le (2MQ^2)^{1/2}\int_{M+}^{\infty} \frac{dW^2 \rho_+^{1/2}(W^2)W_L^{1/2}(Q^2,W^2)}{(W+M)[(W-M)^2+Q^2]^{1/2}} \tag{10a}$$

$$\le (2MQ^2)^{1/2}\int_{1+}^{\infty}\frac{d\omega}{\omega^{1/2}(\omega-1)^{1/2}}\rho_+^{1/2}[Q^2(\omega-1)]F_L^{1/2}(Q^2,\omega). \tag{10b}$$

In writing (10b) we have transformed to the scaling variable $\omega = 2M\nu/Q^2$ and, in anticipation of taking the large-Q^2 limit, have suppressed terms of $O(M^2/Q^2)$. In an asymptotically free theory both the large-Q^2 behavior of $F_L(Q^2,\omega)$ *and* the large-W^2 behavior of $\rho_+(W^2)$ are calculable. The former is, of course, well known and vanishes like $(\log Q^2)^{-1}$ faster than the conventional transverse piece.[6] The behavior of ρ_+ is less well known; however, in axial gauges a straightforward calculation shows $W^2\rho_+(W^2) \to (\ln W^2)^{-(1+\epsilon)}$, where $\epsilon > 0$, as $W^2 \to \infty$. Putting these together, one concludes that the right-hand side of (10b) vanishes logarithmically as $Q^2 \to \infty$. This can be seen in a slightly different way by applying the Schwarz inequality to the integral itself to obtain

$$|1 - G(Q^2)|^2 \le (Z_2^{-1} - 1)\int_0^1 dx\,\frac{F_L(Q^2,x)}{1-x}, \tag{11}$$

where[2] $\int dW^2\rho_+(W^2) \equiv Z_2^{-1} - 1$.

In the axial gauge used here, Z_2 is finite. The integral over F_L, on the other hand, vanishes for large Q^2 so that the right-hand side of (11), and consequently (10), also vanishes. *The left-hand side, however, does not in general vanish;* only if the "elastic" form factor $G(Q^2) \to 1$ as $Q^2 \to \infty$ would this happen. Indeed most investigations[7] have found that $G_E(Q^2) \to \exp(-\ln^2 Q^2)$ and this can be readily extended to $G(Q^2)$ itself. We have thus arrived at a contradiction. The crucial assumption made was that quarks can go on shell in the usual way and that this leads to standard dispersion relations in the W plane. One might therefore tentatively conclude that QCD does not allow quarks to be free. However, before doing so we must discuss more precisely what "going on shell in the usual way" means and clarify what this has to do with confinement. It is obvious that the 1 in Eqs. (10) and (11) is critical in allowing an inconsistency to develop. As already mentioned, this is associated with the "static charge" of the quark. However, a further crucial ingredient is the presumed existence of the matrix element $\langle 0|\psi|p\rangle$. To see how this is related to the singularity structure of the propagator, consider the generalized vertex where *both* fermions are off shell:

$$\tilde{G}_\mu(p,p') = \int d^4x \int d^4y\, e^{iq\cdot x + ip'\cdot y}\langle 0|T[\bar{\psi}(y)j_\mu(x)\psi(0)]|0\rangle. \quad (12)$$

According to the conventional LSZ prescription,[1] the original vertex, (1), is given by $G_\mu(p,p') = \lim_{\not{p}\to M}(\not{p}-M)\tilde{G}_\mu(p,p')$. Now, the Ward identity (W.I.) for $\tilde{G}_\mu$ reads

$$q^\mu \tilde{G}_\mu(p,p') = S_F(p) - S_F(p'). \quad (13)$$

Thus,

$$q^\mu G_\mu(p,p') = \lim_{\not{p}\to M}(\not{p}-M)S_F(p) \quad (14)$$

which agrees with Eq. (3) only if $S_F(p)$ has the canonical $(\not{p}-M)^{-1}$ singularity when $\not{p}\to M$. The factor 1 in Eq. (9) therefore represents, not only the static charge, but also the normalization resulting from the cancellation of this pole by its inverse. We can therefore interpret the inconsistency as a statement that, in QCD, the quark does not go on shell via the standard simple-pole singularity prescription. This, of course, is hardly surprising since the masslessness of the gluons inevitably gives rise to an infinite number of thresholds beginning at $W = M$ and these presumably alter the structure of any LSZ formulation. In quantum electrodynamics (QED) we know precisely what happens since the infrared behavior is calculable: a cut develops. Typically, the propagator behaves like $S_F{}^0(p) \equiv (\not{p}-M)(p^2 - M^2)^{-(1+A\alpha)}$ when $\not{p}\to M$ where A is a calculable number.[8] The fact that $S_F{}^0(p)$ is not a simple pole does *not*, of course, mean that electrons cannot be detected. The miracle of QED is that one can cut an external electron line in such a way that it takes with it the right infinite combination of photons to make a real detectable electron. Formally this can be expressed in terms of a generalization of the conventional LSZ formalism[8]: define a Green's function analogous to Eq. (12)

$$G(p_1,p_2,\ldots,p_n) \equiv \int d^4x_1 \int d^4x_2 \cdots \int d^4x_{n-1} \exp(i\textstyle\sum p_i\cdot x_i)\langle 0|T[\psi(x_{n-1})\cdots\psi(x_2)\psi(x_1)]|0\rangle; \quad (15)$$

then the corresponding "S-matrix element" is

$$S(p_1,\ldots,p_n) = \prod_{i=1}^{n} \lim_{\not{p}_i\to M_i}[S_F{}^0(p_i)]^{-1} G(p_1,\ldots,p_n). \quad (16)$$

Apart from singular interparticle Coulomb phases, the external mass singularities in G are precisely cancelled by the inverse propagators leading to a finite gauge invariant S from which the physical cross section can be calculated.[8] We now generalize this in the following way: *We define a theory to be a nonconfining theory if it is possible to derive the S matrix by multiplying the corresponding Green's function by the inverse propagators of the external fields taken one at a time in the limit as each field approaches its mass shell. A confining theory is defined to be one where this is not possible.* In general, one expects this definition to be augmented by singular interparticle phase factors for each particle as it goes on shell. However, for the case considered here, namely G_μ of Eq. (1), there is only *one* on-shell particle so the question of interparticle singular phases is irrelevant. Let us suppose then that when $\not{p} \to M$, $S_F(p) \to S_F{}^0(p)$ which in QCD is unknown. If the theory is nonconfining the vertex analogous to G_μ of Eqs. (1) and (13) is $G_\mu'(p,p') \equiv \lim_{\not{p}\to M}[S_F{}^0(p)]^{-1}G_\mu(p,p')$ and, by assumption, this exists. The W.I. for G_μ' reads

$$q^\mu G_\mu'(p,p') = \lim_{\not{p}\to M}[S_F{}^0(p)]^{-1}[S_F(p) - S_F(p')] \quad (17)$$

which equals 1 provided $\lim_{\not{p}\to M}[S_F{}^0(p)]^{-1}$ vanishes. It is obviously most convenient to consider $\Gamma_\mu \equiv (\not{p}'-M)G_\mu'$, as before, since this preserves the form of our previous equations.[9] We therefore see that the "1" that occurs in the inequality is simply a reflection of the "1" embedded in the generalized W.I., Eq. (17). Thus, the inconsistency that resulted before remains valid regardless of the structure of $S_F{}^0(p)$. We therefore conclude that QCD does not allow quarks to go on shell and, consequently, that it confines! Some remarks are in order:

(i) The calculation was performed in an axial gauge. An entirely different gauge for this problem is the projected Lorentz gauge of Oehme and Zimmerman[10] in which only the positive definite part of the Hilbert space is used. An identical result follows; indeed they prove that, in their "gauge", not only is $S_F(p')$ analytic in W, but that Z_2 is finite.

(ii) The form of the basic inequality is model independent. The critical ingredient from QCD is that asymptotic freedom ensures that both $F_L(Q^2, x)$ and $\rho_+(W)$ vanish sufficiently fast asymptotically. Note that, since neither QCD nor standard spontaneously broken gauge theories are asymptotically free, no inconsistencies arise there. Furthermore, it is not difficult to check that the inequality is satisfied in the infrared (i.e., in perturbation theory) since the crucial "1" on the left-hand side is canceled by $G(Q^2) \to 1 + O(g^2)$, as $g \to 0$. Thus, as expected, confinement is not a property of perturbation theory.

(iii) In drawing conclusions from Eqs. (10), we have apparently made no reference to the explicit particle under discussion. However, had it been a nucleon, for example, its field would be at least cubic in ψ. Ordinary dimensional arguments would then require $\rho_+(W^2)$ to be highly divergent for large W^2 allowing the inequality to be well satisfied, and nucleons to be free.

(iv) Finally, we note that one can use any conserved current occurring in the theory to generate this result. Thus, for example, one can extend the technique to show that gluons are confined by using the energy-momentum tensor. This, and various other aspects of the problem will be explored in a later paper.

[1]For a review of the conventional Lehmann-Symanzik-Zimmermann (LSZ) formulation see, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Chap. 16.

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[7]C. P. Korthals Altes and E. de Rafael, Nucl. Phys. B106, 237 (1976).

[8]D. Zwanziger, Phys. Rev. D 11, 3481 (1975), and references contained therein. See also Ref. 1, Chaps. 17 and 19.

[9]It is possible to use other definitions of Γ_μ such as $[S_F(p')]^{-1}G_\mu'$ or G_μ' to derive the desired result.

[10]R. Oehme and W. Zimmermann, Phys. Rev. D 21, 471 (1980).