Importance of Coulomb Effects in Half-Shell Scattering

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The half-shell cross section for *charged*-particle scattering is discontinuous at the on-shell point. Moreover, its limits for $p \to k^-$ and $p \to k^+$ not only differ from each other, they also differ from the corresponding on-shell cross section, often by appreciable energy-dependent factors. This has important consequences for the theoretical description of bremsstrahlung and quasifree data.

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The description of many scattering experiments involving composites of charged and neutral particles can be given in terms of half- (on-the-energy-) shell scattering amplitudes. Examples of such processes are knockout reactions and other quasifree processes in particle and nuclear physics. Also in bremsstrahlung processes half-shell scattering occurs, and the primary goal of many bremsstrahlung experiments has been to obtain information on the off-shell behavior of scattering amplitudes, or transition (T) matrices. In the case of *short-range* interactions the off-shell T matrix is a continuous function of the off-shell momenta at their on-shell value. For interactions with a r^{-1} tail at large distances (*charged* particles) this is no longer the case. This fact and its consequences have hitherto not received the attention they deserve. They constitute the subject of this Letter.

It is well known that, in the case of charged particles, the long range of the Coulomb interaction is a source of special difficulties. One of the effects of the long range is that the (physical) half-shell and the off-shell Coulomb T matrices have (branch-point) singularities at the on-shell value of the physical half-shell and off-shell variables. These singularities lead to physically observable effects. We adopt notations and conventions developed and used previously.[1,2] A convenient framework is provided by so-called (Coulombic) asymptotic states $|\vec{k}\infty\pm\rangle$. Often we suppress the + symbol in this notation. We consider scattering by a potential $V = V_C + V_s$, where V_s is a short-range potential, and $V_C(r) = Ze^2/r = 2k\gamma/r$ is the Coulomb potential. We set $\hbar = 1$ and $2m = 1$, so that $E = k^2$ is the energy, and γ is Sommerfeld's parameter. The T operator can be split into a pure Coulomb part T_C and a remainder T_{cs}, $T = T_C + T_{cs}$. The on-shell and half-shell amplitudes are connected to the physical on-shell T matrix ($k = k'$) and the physical half-shell T matrix through[1,2]

$$-(2\pi^2)^{-1} f_{\rm on}(\hat{k}\cdot\hat{k}') = \langle \vec{k}'\infty - | T | \vec{k}\infty\rangle, \qquad \hat{k}' \neq \hat{k}, \quad k' = k \in R^+, \tag{1}$$

$$-(2\pi^2)^{-1} f_{\rm half} = \langle \vec{p} | T | \vec{k}\infty\rangle, \quad p \neq k, \tag{2}$$

respectively. The argument of the T operator is $(k+i\epsilon)^2$, $\epsilon \to 0^+$. Often we write k for $k+i\epsilon$, $\epsilon \to 0^+$. For the pure Coulomb T operator T_C the physical on-shell and half-shell elements are known explicitly,[1,2]

$$-(2\pi^2)^{-1} f^{C}(\hat{k}\cdot\hat{k}') = \langle \vec{k}'\infty - | T_C | \vec{k}\infty\rangle = \frac{k\gamma}{\pi^2 Q^2}\exp(2i\sigma_0)\left(\frac{4k^2}{Q^2}\right)^{i\gamma}, \tag{3}$$

$$-(2\pi^2)^{-1} f_{\rm half}{}^{C} = \langle \vec{p} | T_C | \vec{k}\infty\rangle = C_0 \exp(i\sigma_0)\frac{k\gamma}{\pi^2 q^2}\left(\frac{p^2-k^2}{q^2}\right)^{i\gamma}, \tag{4}$$

where f^C is the Coulomb amplitude, $\vec{Q}$ and $\vec{q}$ are momentum transfers,

$$\vec{Q} = \vec{k}' - \vec{k}, \quad \vec{q} = \vec{p} - \vec{k}. \tag{5}$$

Throughout we take $\hat{p} = \hat{k}'$, so that $\lim_{p\to k}\vec{q} = \vec{Q}$. The Coulomb phase shift is $\sigma_0 = \arg\Gamma(1+i\gamma)$, the Coulomb penetrability is $C_0{}^2 = 2\pi\gamma/[\exp(2\pi\gamma) - 1]$. Amplitudes and cross sections are related through $\sigma(x) = f^* f$. For the pure Coulomb case Eqs. (3) and (4) give

$$\sigma_{\rm half}{}^{\rm Coul}(\hat{p}\cdot\hat{k}) = \sigma_{\rm on}{}^{\rm Coul}(\hat{k}'\cdot\hat{k})C_0{}^2 \mathscr{E}^2 Q^4/q^4, \tag{6}$$

where $\mathcal{S}$ (for $p \neq k$) is defined by

$$\mathcal{S} = \begin{cases} 1 \text{ for } p > k, \\ \exp(\pi\gamma) \text{ for } p < k. \end{cases} \quad (7)$$

Equations (3) and (4) show that the (on-shell) limits for $p \to k^-$ and $p \to k^+$ of the physical half-shell T matrix element $\langle \vec{p} | T | \vec{k} \infty \rangle$ are not equal to the on-shell T matrix element. Both these limits do not exist. Nevertheless, the on-shell limits for $p \to k^-$ and for $p \to k^+$ of the *modulus* of the physical half-shell T matrix (of the half-shell cross section) *do* exist. However, these two limits differ from each other, and both differ from the modulus of the physical on-shell T matrix (from the on-shell cross section), according to

$$\lim_{p \to k^+} \sigma_{\text{half}} = \exp(-2\pi\gamma) \lim_{p \to k^-} \sigma_{\text{half}} = C_0^2 \sigma_{\text{on}}. \quad (8)$$

For a repulsive Coulomb potential ($k\gamma > 0$) the $p \to k^+$ limit of σ_{half} is smaller than σ_{on}, whereas the $p \to k^-$ limit is larger than σ_{on}.

In the pure Coulomb case Eq. (6) shows that the ratio $\sigma_{\text{half}}/\sigma_{\text{on}}$ consists of two factors. Only the first one, $C_0^2\mathcal{S}^2$, survives in the on-shell limit, cf. Eq. (8). For $p \neq k$ it is independent of p, i.e., it is not dependent on how far one is off shell. Instead, it is highly energy dependent through $\gamma \equiv \frac{1}{2}Ze^2/k$. The other factor, Q^4/q^4, lies between 0 and $(\frac{1}{2} + \frac{1}{2}p/k)^{-4}$ for all $p, k > 0$. For fixed $\hat{k} \cdot \hat{k}'$ it depends on the off-shell variable p/k only. For $p > k$ it represents a suppression factor (between 0 and 1) which is particularly effective in forward directions. Note that if we consider the cross sections as functions of energy and *momentum transfer* (instead of energy and *scattering angle*), the following relation between $\sigma_{\text{half}}^{\text{Coul}}(E, q^2)$ and $\sigma_{\text{on}}^{\text{Coul}}(E, Q^2)$ holds:

$$\sigma_{\text{half}}^{\text{Coul}}(E, q^2) = \sigma_{\text{on}}^{\text{Coul}}(E, q^2) C_0^2 \mathcal{S}^2. \quad (9)$$

Note also that in half-shell scattering the magnitude of the momentum transfer $q = |\vec{p} - \vec{k}|$ can take values larger than $2k$, if $p > k$. Such large values of q are inaccessible in on-shell scattering.

In principle Eqs. (6) and (8) for the pure Coulomb potential have been known for twenty years already.[1-3] In this Letter we wish to point out that Eq. (8) holds *true for any potential* $V_C + V_s$. The mathematical proof of this is simple. For $k = k'$,

$$T_{cs} | \vec{k}' \infty \rangle = (1 + T_C G_0) t_{cs} | \vec{k}'^+ \rangle_C, \quad (10)$$

where the operator t_{cs} satisfies

$$t_{cs}(E) = V_s + V_s G_C(E) t_{cs}(E),$$

$|\vec{k}+\rangle_C$ are Coulomb scattering states, and G_0 and G_C are the usual free and Coulomb resolvents, respectively. Let $\int d^3p \, \langle \vec{k} \infty - | \vec{p} \rangle \langle \vec{p} |$ operate on both sides of Eq. (10). Upon using[1,2]

$$\langle \vec{k} \infty - | \vec{p} \rangle = \langle \vec{p} | \vec{k} \infty \rangle = \delta(p, k) \lim_{\epsilon \to 0^+} \left(\frac{p - k - i\epsilon}{p + k} \right)^{-i\gamma} \frac{e^{\pi\gamma/2}}{\Gamma(1 - i\gamma)},$$

we find

$$\lim_{p \to k} \lim_{\epsilon \to 0^+} \left(\frac{p - k - i\epsilon}{p + k} \right)^{-i\gamma} \left(\frac{e^{\pi\gamma/2}}{\Gamma(1 - i\gamma)} \right) \langle \vec{p} | T | \vec{k}' \infty \rangle = \langle \vec{k} \infty - | T | \vec{k}' \infty \rangle,$$

so that

$$\lim_{p \to k} \left(\frac{p - k}{p + k} \right)^{-i\gamma} \langle \vec{p} | T | \vec{k}' \infty \rangle = C_0 \exp(-i\sigma_0) \langle \vec{k} \infty - | T | \vec{k}' \infty \rangle. \quad (11)$$

By virtue of Eqs. (1) and (2) this proves Eqs. (8) and (9) for potentials $V_C + V_s$.

The physical ingredient in this proof is that it is only the Coulomb tail of the potential which determines the singular behavior of the half-shell cross section as given in Eqs. (8) and (9). It is instructive to consider the case that V_s is a rank-one potential in the partial-wave space characterized by l,

$$V_{s,l} = -\lambda_l | g_l \rangle \langle g_l |, \quad (12)$$

where the form factor g is taken of a simple rational form

$$\langle p | g_l \rangle \equiv g(p) = (2/\pi)^{1/2} p^l (p^2 + \beta^2)^{-l-1}. \quad (13)$$

For $l = 0$ (Yamaguchi potential) and for $l = 1$ closed analytical expressions for f_{on} and f_{half} are given in the existing literature.[1,2] Recently we have succeeded for arbitrary l to calculate the amplitudes (1) and (2) in closed form, too.[4] Obviously they satisfy Eq. (11), and hence Eq. (8).

For the Yamaguchi case we show in Fig. 1(a) the ratio $\sigma_{\text{half}}(E, q^2)/\sigma_{\text{on}}(E, q^2)$ for seven E values as a function of q. As an illustrative example we have chosen the parameters Ze^2, λ, and β to fit low-energy $^1S_0$ proton-proton scattering data, $\lambda = 2.4 \text{ fm}^{-3}$ and $\beta = 1.1 \text{ fm}^{-1}$. No Pauli symmetrization was carried out. For the same E values Fig. 1(b) gives the ratio $\sigma_{\text{half}}(E, \theta)/\sigma_{\text{on}}(E, \theta)$ as a function of θ, with $\cos\theta = \hat{p} \cdot \hat{k}$. Curves are labeled from 1 to 6, corresponding to the values of the half-shell variable $p/k = 0.8$, 0.9, 0.999, 1.001,

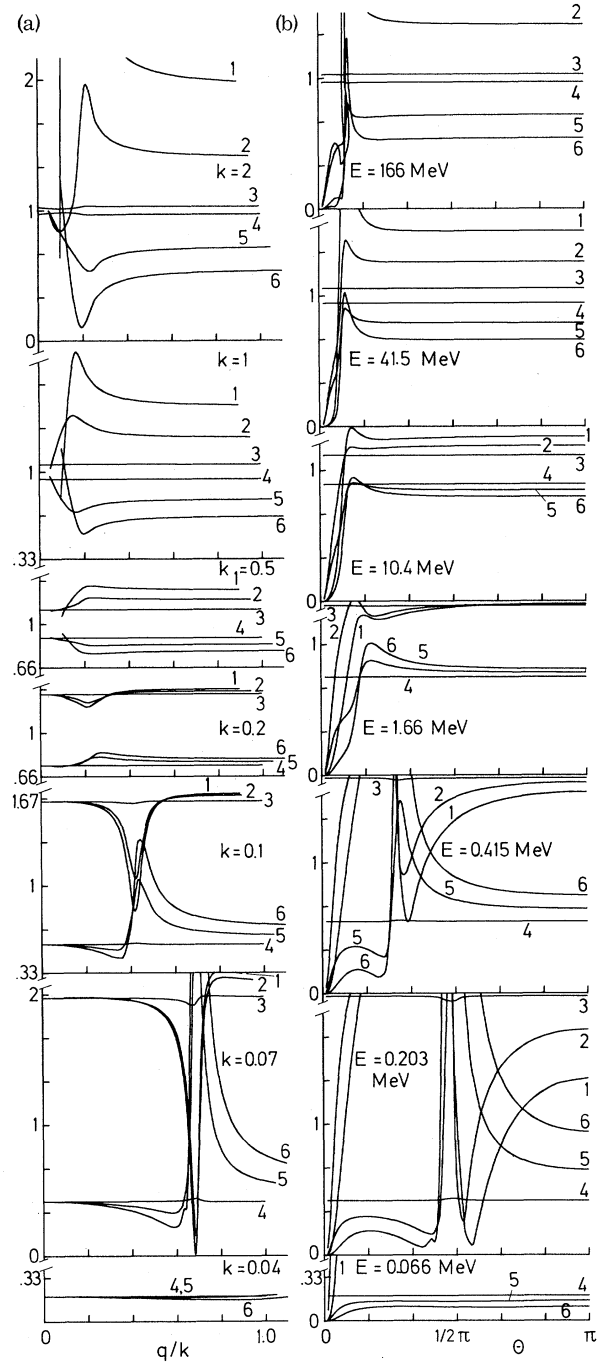

FIG. 1. (a) The half-shell ratio $\sigma_{\text{half}}(E,q^2)/\sigma_{\text{on}}(E,q^2)$ vs q/k and (b) $\sigma_{\text{half}}(E,\cos\theta)/\sigma_{\text{on}}(E,\cos\theta)$ vs θ, for proton-proton ${}^1S_0$ parameters $Z=1$, $\lambda=2.4\ \text{fm}^{-3}$, $\beta=1.1\ \text{fm}^{-1}$, without Pauli symmetrization. Plots for seven energies $E=k^2$ are given; from bottom to top $k=0.04$, 0.07, 0.1, 0.2, 0.5, 1, and 2 fm^{-1}, so that in the center-of-mass system $E=0.664$, 0.203, 0.415, 1.66, 10.4, 41.5, and 166 MeV, and $\gamma=0.43$, 0.25, 0.174, 0.087, 0.035, 0.017, and 0.0087, respectively. In each plot the labels 1–6 correspond to values of the half-shell variable $p/k=0.8$, 0.9, 0.999, 1.001, 1.1, and 1.2, respectively.

1.1, and 1.2, respectively. All curves 3 ($p/k=0.999$, nearly on-shell) are practically straight lines at $C_0^2\exp(2\pi\gamma)$, in agreement with Eq. (8). All curves 4 ($p/k=1.001$, nearly on-shell) are practically straight lines at C_0^2, also in agreement with Eq. (8). The curves 3 and 4 lie far apart when γ is large (large charges, low energies). Curves 1, 2, 5, and 6 correspond to cases further off shell. At very low energies the Coulomb force dominates at all momentum transfers, and the half-shell ratio is given essentially by $C_0^2\mathfrak{G}^2$ [Fig. 1(a)], or $C_0^2\mathfrak{G}^2Q^4/q^4$ [Fig. 1(b)]. In these cases suppression of σ_{half} compared to σ_{on} is enormous, when $p>k$. For $p<k$ we have an enhancement. (Note that when the signs of the charges are opposite, there is suppression for $p<k$, and enhancement for $p>k$.) At somewhat higher energies the Coulomb effects still dominate for low momentum transfer. For some intermediate value of q/k (or θ) strong fluctuations occur as a result of interference effects. For larger q (or θ) the half-shell ratio again is more or less constant. In most cases it differs from the value 1 more than $C_0^2\mathfrak{G}^2$ (which gives the ratio in the on-shell limit).

Only at high energies does the influence of Coulomb effects become small for most momentum transfers. The ratio is close to the ratio corresponding to the Coulomb potential switched off ($V_C=0$) only at the highest energies, and only at large values of q. In our example the half-shell ratio for $V_C=0$ is trivially $g^2(k)/g^2(p)$. In Fig. 1(b) the strong suppression near $\theta=0$ due to the suppression factor Q^4/q^4 [cf. Eq. (6)] is manifest in all cases. Also in the region of interference the behavior of the curves in Fig. 1(a) and of those in Fig. 1(b) is rather different.

In the proton-proton system the Coulomb potential is weaker than in many other nuclear systems. Therefore in other systems Coulomb and interference effects may be expected to be even more important. In the description of many quasifree (QF) processes σ_{half} for the process $a+b\to a+b$, or $A+b\to A+b$, enters the expression for the differential cross section for a knockout or breakup experiment $A(a,ab)B$, or $A(c,ab)B$, as a factor.[5,6] For example both in plane-wave impulse approximation and in distorted-wave impulse approximation

$$\frac{d^3\sigma}{d\Omega_a\,d\Omega_b\,dE}=N_{\text{eff}}\cdot(\text{PSF})\cdot(\text{MD})\cdot\left.\frac{d\sigma}{d\Omega}\right|_{\text{half}},\qquad(14)$$

where the phase-space factor (PSF) is known, and (MD) is a factor containing a (possibly distorted) momentum distribution and spectroscopic factors. In many cases the shape of the left-hand side of (14) is well described by the theory, whereas the absolute magnitude is not. To obtain agreement between theory and experiment, one introduces a renormalization factor $N_{\rm eff}$. Often $\sigma_{\rm half}$ is assumed to be equal to $\sigma_{\rm on}$, which then is taken from two-body experiments. This assumption, as we described in this Letter, may be wrong. How wrong depends on the following: (i) How far the half-shell momentum p is off shell. For $p > k$, but p close to k, Eqs. (6)–(9) suggest that the error will be a p-independent factor C_0^2. (ii) To what extent the Coulomb interaction (or interference) dominates the mechanism of the QF process. In forward directions, and at low energies, the Coulomb interaction does dominate, even when p is far off shell. Obviously C_0^2 can give suppression by orders of magnitude!

The factor $C_0^2 \vartheta^2$ has, to our knowledge, received little attention. It is p independent (if $p \neq k$), and strongly energy dependent. In fact, it has a behavior similar to the energy dependence of $N_{\rm eff}$ observed empirically[7]: For repulsive interactions $N_{\rm eff}$ decreases rapidly when k decreases. In bremsstrahlung a similar effective suppression factor is observed with the same type of energy dependence.[8] For large γ we suggest that the analysis of QF data (i) include for $p > k$ the factor C_0^2 in the replacement of $\sigma_{\rm half}$ by $\sigma_{\rm on}$ [cf. Eqs. (9) and (14)], and thereby (ii) take the on-shell cross section at the *same momentum transfer* as the half-shell cross section, and *not at the same scattering angle* [cf. Eqs. (6) and (9), and Fig. 1]. Thereby not only is the normalization of theoretical spectra affected, but also (to a much lesser degree) their shape. In a practical example this recently proved successful.[5] We expect it to be important in the theoretical description of more of the wealth of experimental QF data. It will also be an improvement in the means for obtaining spectroscopic information. It remains interesting to investigate the half-shell cross-section enhancement for these cases in which $C_0^2 \vartheta^2 > 1$.

We note that expanding half-shell cross sections around their on-shell point, as is common in theoretical descriptions of bremsstrahlung experiments, deserves the utmost care.

We close with a remark on the physical consequence of the discontinuity Eqs. (4), (8), and (9). In the transition from time-dependent to time-independent scattering theory one constructs physically meaningful wave-packet averages. Thus the half-shell T matrix has to be integrated over a momentum space distribution. In view of Eq. (4) one should consider $\int_{k_0}^{k_1} f(k)(p-k-i\epsilon)^{i\gamma(k)} \times dk$, where $f(k)$ is a smooth function and $k_1 - k_0$ is small. When p lies outside the integration interval (k_0, k_1), the argument remains unchanged: The modulus of the integrated amplitude has a discontinuity with the same jump as discussed before. When p does belong to the interval (k_0, k_1) we get a smearing effect. The net effect of wave-packet averaging will be that the jump becomes a quasijump. This is similar to the effect of screening of the pure Coulomb potential. However, the overall jump remains the same, i.e., if one is not too close on shell, the cross section will show a jump, given essentially by the quantity ϑ defined by Eq. (7). Integration with respect to p is quite similar. Finally, integration with respect to the angle between $\vec{p}$ and $\vec{k}$ has no effect on the discontinuity.

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[6]Justification of such description has to be provided by rigorous n-particle theory for charged particles. This is extremely difficult, and so far it has not been done.

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