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Lower Critical Dimension and the Roughening Transition of the Random-Field Ising Model

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It is shown by a calculation analogous to that carried out by Wallace and Zia for the pure Ising model that the lower critical dimension of the Ising model in a random field is 3 and not 2 as suggested by domain energy arguments. Further, the critical dimension for the roughening transition is shown to be 5 as compared to 3 for the pure Ising model.

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In a recent letter Wallace and Zia[1] have argued that capillary waves which describe large distance deviations from planar of an essentially sharp interface can be interpreted as the Goldstone modes whose fluctuations lower the critical temperature to zero as $d \to 1^+$. The capillary interface waves therefore play the same role for a system with discrete symmetry as spin waves for a system with continuous internal symmetry. A renormalization-group calculation was carried out in $d = 1 + \epsilon$ with $T_c \sim \epsilon$, showing that the lower critical dimension is 1 for the discrete Ising model.

In this Letter the effect of adding a random field to this model is studied. The lower critical dimension is found to be 3, and the results of a $d = 3 + \epsilon$ calculation are presented. Further, we show that the roughening transition predicted to take place for $d = 3$ for the pure Ising model[2] is shifted to $d = 5$. Thus the interface is rough below the ordering temperature for $3 < d < 5$.

These results are in agreement with a recent letter by Parisi and Sourlas,[3] who argue on very general grounds that for all properties the relevant dimension for the random-field model should increase by 2 as compared with the corresponding property of the pure Ising model. A lower critical dimension of 3 is in apparent disagreement with $T = 0$ domain energy arguments[4] which yield a domain instability below $d = 2$. This has generally been interpreted as a lower critical dimension of 2. However, such an instability provides only a lower bound on the lower critical dimension. In fact, qualitative arguments[5] suggest that considering the domain-wall *free* energy strengthens the instability and moves the lower critical dimension to $d = 3$.

The predicted lack of long-range order for $d = 3$ is consistent with recent experimental studies[6] of disordered $GdAlO_3$ whose antiferromagnetic to paramagnetic transition in a uniform field is similar to that of the Ising model in a random field.[4,7] It is also in agreement with the results[8] on Fe_{1-x}-Co_xCl_2 where the smearing of the transition of one of the order parameters may be due to effective random fields associated with the other order parameter.[8] However, other interpretations are also possible.[9]

The rather common situation of a first-order transition in a system with quenched "T_c" impurities[10] is similar to the transition in the random-field Ising model as a function of a uniform magnetic field below the critical temperature.[10] Thus, our results imply that first-order transitions should always be smeared in impure three-dimensional (3D) systems, if the impurities can be regarded as "quenched." Moreover, for $d=3$ (<5), such systems should display a substantial interface roughening. In fact, for an interface of linear dimension L an effective interface fluctuation of $O(L)$ and proportional[11] to the square root of the impurity concentration is predicted. This impurity-induced interface roughness should be observable.

Various models have been used to describe the interface of spin-up and spin-down domains of the discrete Ising model. Some of these will be discussed below and then generalized to include the effect of the random field.

The capillary-wave approach used by Wallace and Zia[1] gives rise to a reduced Hamiltonian for a field f which describes the deviation of a sharp interface from planar,

$$\mathcal{H}(f)=T^{-1}\int d^{d-1}x\{[1+(\nabla f)^2]^2+\tfrac{1}{2}m^2f^2\}. \quad (1)$$

The first term is the surface area of the interface. The coefficient T^{-1} represents σ/k_BT, where σ is the interfacial energy per unit area at zero temperature. The mass term represents a pinning potential such as gravity for the liquid-solid interface or a step function magnetic field for the discrete Ising model. This model gives rise to an infinite interface width[12] in the limit $m\to 0$ for $d\leqslant 3$. For the 2D Ising model this is in agreement with rigorous calculations,[13] whereas the 3D Ising model has a finite interface width at low temperatures.[14] In 3D the discreteness of the lattice gives rise to a finite interface width below a roughening transition tempersture T_R. For $T_R<T<T_c$ the width is again infinite, as in the continuum models using capillary waves. For $d>3$, $T_R=T_c$ and the interface width diverges as $T\to T_c$.[15,16]

The roughening transition has been described by various solid-on-solid models.[2] A simple such model is the discrete Gaussian (DG) model introduced by Chui and Weeks[2,17] in which the interaction between nearest-neighbor columns is quadratic in the height difference,

$$H_{\rm DG}=\frac{J}{2}\sum_{j,\delta}(h_j-h_{j+\delta})^2\equiv\frac{J}{2}\sum_{j,j'}h_jG^{-1}(j,j')h_{j'}, \quad (2)$$

where $G^{-1}(q)\sim q^2+O(q^4)$ and where h_j are integers.

Following Chui and Weeks[2,17] the partition function may be written $Z_{\rm DG}=Z_0Z_{\rm C}$, where Z_0 is the partition function of the continuous Gaussian model, equivalent to the capillary-wave approximation, and can be evaluated exactly, and where $Z_{\rm C}$ is the partition function of the neutral Coulomb gas,

$$Z_{\rm C}=\sum_{\{k_j\}=-\infty}^{\infty}\exp\left[-\frac{k_BT}{2J}\sum_{j,j'}k_jG(j,j')k_{j'}\right], \quad (3)$$

in which k_j represents the charges. Note the q^{-2} dependence of $G(q)$ at small q which characterizes the Coulomb interaction. In this way the roughening transition has been related to the metal-insulator transition of the Coulomb gas, which is in turn related to the Kosterlitz-Thouless transition in the planar XY model.[18,19]

The reduced temperature k_BT/J has been inverted in going from the DG model to the Coulomb gas. Thus while in the XY model the spin-wave excitations persist[18] (although with renormalized stiffness) in the presence of the bound vortex-antivortex pairs for $T<T_{\rm KT}$, the gapless capillary waves persist in the presence of the discrete lattice (Coulomb gas) for $T>T_R$. Because the interface has dimension $d-1$, a finite nonzero value of T_R is thus obtained for the 3D Ising model. For $d>3$, $T_R=T_c$ while for $d<3$, $T_R=0$.[20] Thus for $d<3$ the capillary wave approximation should be applicable for $T\gtrsim 0$. It was used, in the form given by Eq. (1), by Wallace and Zia in their renormalization-group calculation in $1+\epsilon$ dimensions.

Wallace and Zia did not establish a rigorous connection between the transition obtained from the model, Eq. (1), in terms of the interface variable f and the transition of the underlying Ising model. However, from their interface model they find that the surface tension vanishes for $T\to T_c^-$ in the way predicted by the scaling law for the Ising model,[15] and in agreement with exact results for the 2D Ising model.[21,22] Further, their results for the correlation length of the 1D Ising model agree with exact results. This supports the identification of T_c as obtained from the interface model Eq. (1) with the T_c of the Ising model. For the 2D Ising model this connection has been made more explicit by Einhorn and Savit.[23] They show that the phase transition corresponds to a condensation of the domain boundaries. More generally, in the paramagnetic phase of the Ising model just above T_c, clusters with sizes comparable to the correlation length ξ are expected to exist. This agrees with having the ff correlation function of the interface decay over a length ξ.

Still another approach used to describe the interface starts from the Landau-Ginzburg-Wilson free-energy functional,[16]

$$\mathcal{H}(s)=\int d^dx[\tfrac{1}{2}(\nabla s)^2+\tfrac{1}{2}rs^2+us^4]\,. \tag{4}$$

Following Ref. 16, we set $s(x)=M(x)+\sigma(x)$, where $M(x)$ is at this stage arbitrary. The free energy is a functional of $M(x)$

$$F[M(x)]=-k_BT\ln\mathrm{Tr}_\sigma e^{-\mathcal{H}[M+\sigma]}. \tag{5}$$

If $\sigma(x)$ is restricted such that $\langle\sigma(x)\rangle_M=0$, the equation of state follows from $\delta F/\delta M(x)=0$. If fluctuations involving σ are neglected, the Landau-Ginzburg equation is recovered, $-\nabla^2M+rM+4uM^3=0$, with solution $M_0(z)+(-r/4u)^{1/2}\tanh(-\frac{1}{2}\times r)^{1/2}z$, for which variation is assumed in only one direction. Inclusion of quadratic fluctuations about this solution gives

$$\mathcal{H}(\sigma)=\int d^dx[\tfrac{1}{2}(\nabla\sigma)^2+\tfrac{1}{2}r\sigma^2+6uM_0^2(z)\sigma^2]. \tag{6}$$

The eigenvalues and eigenvectors of this effective Hamiltonian are obtained from the differential equation, $-\nabla^2\psi+r\psi+12uM_0^2(z)\psi=\lambda\psi$. All the eigenvalues $\lambda_n(q,p)$ and eigenfunctions $u_n(q,p)$ of this equation are known. Here q is a $(d-1)$-dimensional wave vector associated with the transverse degrees of freedom and p is a scalar variable associated with z. If we keep only the lowest eigenvalue of the z motion, which is gapless and proportional to q^2, we recover the capillary-wave approximation (or continuous Gaussian model) discussed above.

This last approach is readily generalized to include the effect of the random field with the help of the replica method. We add a random field to the effective Hamiltonian [Eq. (4)], replicate the partition function, and average over the random field to obtain a new effective Hamiltonian,

$$\mathcal{H}[s]=\int d^dx\{\sum_\alpha[\tfrac{1}{2}(\nabla s_\alpha)^2+\tfrac{1}{2}rs_\alpha^2+us_\alpha^4]-\tfrac{1}{2}\tilde{\Delta}\sum_{\alpha,\beta}s_\alpha s_\beta\}, \tag{7}$$

where we have assumed a Gaussian distribution of width $\tilde{\Delta}$, and where α and β are replica indices. We set $s_\alpha(x)=M(x)+\sigma_\alpha(x)$, where $\langle\sigma_\alpha\rangle_M=0$. M is independent of the replica and is given by the same expression as for the pure case. The quadratic fluctuations about this solution now take the form

$$\mathcal{H}[\sigma]=\int d^dx\{\sum_\alpha[\tfrac{1}{2}(\nabla\sigma_\alpha)^2+\tfrac{1}{2}r\sigma_\alpha^2+6uM_0^2(z)\sigma_\alpha^2]-\tfrac{1}{2}\tilde{\Delta}\sum_{\alpha,\beta}\sigma_\alpha\sigma_\beta\}. \tag{8}$$

For $\tilde{\Delta}=0$, this Hamiltonian can be diagonalized as before,

$$\mathcal{H}=\int d^{d-1}q\,dp\{\sum_{n,\alpha}\lambda_n(p,q)f_n^{\alpha}(q,p)f_n^{\alpha}(-q,-p)-\tfrac{1}{2}\tilde{\Delta}\sum_{\alpha\beta}\sigma_\alpha(p,q)\sigma_\beta(-q,-p)\}. \tag{9}$$

The original fluctuations σ_α can be expressed as linear combinations of the u_n^{α} with coefficients $f_n^{\alpha}(p,q)$. Thus finally, keeping only the lowest-lying mode in the effective Hamiltonian gives,[24,25]

$$\mathcal{H}=\int d^{d-1}q[q^2\sum_\alpha f^{\alpha}(q)f^{\alpha}(-q)-\Delta'\sum_{\alpha,\beta}f_\alpha(q)f_\beta(-q)]. \tag{10}$$

Analogously, in the presence of a random field the model of Wallace and Zia, Eq. (1), is generalized to[24,25]

$$\mathcal{H}(f)=T^{-1}\int d^{d-1}x\Big(\sum_\alpha\{[1+(\nabla f^{\alpha})^2]^{1/2}+\tfrac{1}{2}m^2(f^{\alpha})^2\}-(\Delta/T)\sum_{\alpha,\beta}f^{\alpha}f^{\beta}\Big) \tag{11}$$

while the discrete Gaussian model takes the form[24]

$$Z_{\mathrm{DG}}=\int\prod_\alpha d[h_j^{\alpha}]\prod_j W(h_j^{\alpha})\exp\Big[-\frac{1}{k_BT}\tilde{\mathcal{H}}_{\mathrm{DG}}\Big]; \tag{12}$$

$$\tilde{\mathcal{H}}_{\mathrm{DG}}=\frac{J}{2}\sum_{q,\alpha}|h_q^{\alpha}|^2G^{-1}(q)-\frac{\Delta''}{k_BT}\sum_{q,\alpha,\beta}h_q^{\alpha}h_{-q}^{\beta},$$

$$W(h_j^{\alpha})=\sum_{k_j^{\alpha}=-\infty}\exp(ik_j^{\alpha}h_j^{\alpha}).$$

As for the pure case we can write $Z_{\mathrm{DG}}=Z_0Z_{\mathrm{C}}$ where now,

$$Z_{\mathrm{C}}=\sum_{[k_i^{\alpha}]=-\infty}^{\infty}\exp\Big(\frac{-k_BT}{2J}\sum_{i,j,\alpha,\beta}k_i^{\alpha}G^{\alpha\beta}(i,j)k_j^{\beta}\Big) \tag{13}$$

$$G_{\alpha\beta}(q)=\Big(\frac{\delta_{\alpha\beta}}{q^2}+\frac{\Delta''}{k_BT}\frac{1}{q^4}\Big).$$

Because of the presence of the $1/q^4$ term, a logarithmic interaction between the charges k_i^{α} is now obtained in four interface dimensions (five for the underlying Ising model) rather than in two (three) for the pure case where Δ'' now plays the role of temperature. The classical logarithmic gas undergoes an infinite-order transition in any dimension.[26] Thus in the presence of a random field there is a roughening transition as a function of Δ at a critical dimension which changes from 3 for the pure Ising model to 5 for the random-field Ising model. That is $\Delta_R = 0$ for $d < 5$, $\Delta_R = \Delta_c$ for $d > 5$, and $0 < \Delta_R < \Delta_c$ for $d = 5$. Thus for the random-field model the capillary-wave approximation should be applicable at low temperatures for $d < 5$. This approximation as expressed in Eq. (11) will be used in a renormalization-group calculation in $d = 3 + \epsilon$ dimensions. This calculation is closely analogous to the $d = 1 + \epsilon$ calculation of Wallace and Zia. However, again the transition occurs as a function of Δ. We find $\Delta_c \propto \epsilon$ for $d = 3 + \epsilon$. From Eq. (11), the propagator takes the form

$$\langle f_\alpha(q) f_\beta(q')\rangle = \delta(q + q')[(T/q^2)\delta_{\alpha\beta} + \Delta/q^4]. \tag{14}$$

Again the $1/q^4$ term shifts the relevant dimension by 2. The recursion relations are given by[25]

$$\begin{aligned} d\Delta(l)/dl &= -(d-3)\Delta(l) + \pi^{-1}\Delta^2(l), \\ dT(l)/dl &= -(d-1)T(l) + \tfrac{1}{2}\pi^{-1}\Delta(l)T(l), \end{aligned} \tag{15}$$

with the fixed points (a) $T^* = 0$, $\Delta^* = 0$; and (b) $T^* = 0$, $\Delta^* = \pi(d-3)$. Linearizing the recursion relations about the fixed point (b) determines the exponent ν, $1/\nu = \lambda_\Delta = \epsilon$. Thus for $d = 3$, $\nu = \infty$, and the correlation length diverges as $e^{1/\Delta}$, as $\Delta \to 0$ ($\Delta_c = 0$, $d = 3$). If we define $\Delta_r = \Delta - \Delta_c$ where $\Delta_c = \pi\epsilon$, the free energy satisfies the scaling law $F(T, \Delta_r) \sim \Delta_r^{(d-2)\nu} F(T)$, such that $(d-2)\nu = 2 - \tilde{\alpha}$, when $\tilde{\alpha}$ is the exponent of $\partial^2 F/\partial \Delta_r^2$. Thus for the transition in the Δ variable once again the scaling law is changed by $d \to d - 2$. If instead of $\Delta - \Delta_c(T)$ we consider $T - T_c(\Delta)$, the same critical exponents are expected.

To summarize, by extending recent calculations for the pure Ising model to include the effect of a random field, we have shown that the lower critical dimension and the critical dimension for the roughening transition are both shifted by 2, as compared with the pure Ising model. In particular, the 3D random-field Ising model is found to be disordered at all temperatures, except at $T = 0$, with a correlation length diverging as $e^{1/\Delta}$, as $\Delta \to 0$. As a corollary, first-order transitions with quenched T_c impurities should exhibit smearing in three dimensions. The interfaces for quasi phase equilibrium (which exist only for finite clusters) in such systems should show a substantial roughness.

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the different models discussed. Thus Δ is measured in units of J^2, $\Delta'' \propto J^2\Delta$, etc.

[25]This procedure gives the correct quadratic Hamiltonian in the physically relevant limit $n \to 0$ (n is the replica index). An alternate derivation yields the correct translationally invariant form $\Delta(f_\alpha - f_\beta)^2$ as well as higher-order terms proportional to Δ such as $(f_\alpha - f_\beta)^4$, $(\nabla f_\alpha)^2(f_\alpha - f_\beta)^2$, etc. These terms would be difficult to derive by the procedure used here. All the eigenvalues and eigenvectors of Eq. (6) would have to be taken into account. However, these terms can be shown to be irrelevant, in the renormalization group sense, and will not be included here. The details of the derivation and of the renormalization group calculation will be presented elsewhere (D. Mukamel, Y. Imry, and E. Pytte, to be published).

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Measurement of the Difference in *pn* Total Cross Sections in Pure Longitudinal Spin States

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The total–cross-section difference in pure longitudinal spin states for p-d interactions has been measured at momenta from 1.1 to 6 GeV/c. Spin-dependent Glauber-type corrections and other corrections have been made to obtain $\Delta\sigma_L(pn)$ and $\Delta\sigma_L(I=0)$. These measurements are of fundamental interest and will also help in determining the existence and nature of dibaryon resonances.

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We have measured the total–cross-section difference in pure longitudinal spin states for proton-neutron interactions [$\Delta\sigma_L(pn)$] at eleven momenta from 1.1 to 6.0 GeV/c. A polarized proton beam and polarized deuteron target were employed because polarized neutron beams of the required intensity and energy are not easily produced. These measurements were made at the Argonne National Laboratory zero-gradient synchrotron (ZGS) by the group that made previous similar measurements for proton-proton interactions in pure longitudinal spin states.

The strong energy dependence in $\Delta\sigma_L(pp)$ data,[1] together with data on the transverse-spin-state cross-section difference $\Delta\sigma_T(pp)$,[2] polarization and various two- and three-spin parameters in elastic scattering, some inelastic channels in nucleon-nucleon interactions, and π-d and p-He scattering, have led to interpretations in terms of dibaryon resonances.[3-7] Further measurements in the elastic and inelastic channels are underway at Clinton P. Anderson Meson Physics Facility, TRIUMF, Schweizerisches Institut für Nuklearforschung, and other laboratories, and data taken at the ZGS by our group and others are being analyzed.

There are several approaches to nucleon and subnucleon physics for which total nucleon cross sections in both pure spin states and pure isospin states are of fundamental interest. These include determining nucleon-nucleon scattering amplitudes, gaining information on particular exchanges in the nucleon-nucleon force and on nucleon-nucleon couplings to nucleon-isobar or isobar-isobar channels, and studying possible multiplets of multiquark resonances in order to learn about the constituent interactions.

The existence of isospin-1 dibaryons would suggest the existence of isospin-0 dibaryon states and indeed some bag models and other approaches predict many such states.[5] The isospin-0 channel is of particular interest also because this channel cannot couple to $N\Delta$ or πD channels which have been discussed in connection with the interpretation of the $pp(I=1)$ case near threshold.[8] Furthermore the thresholds for $NN^*(1400)$ and $\Delta\Delta(1236)$ are higher than 1.5 GeV/c (1.7 and 2.1 GeV/c, respectively).

The beam arrangement and general features of the transmission counters and target used for the first running period have been described previously.[1,9] A ^{3}He-^{4}He dilution refrigerator was used for the target during the second running period. The target NMR system could record both