temperature T_i gives the ion density,

$$n_i(\varphi(s)) = n_0 \exp[-e\varphi(s)/kT_i]. \quad (7)$$

By equating the two densities, it is found that the potential barrier for ions is $e\varphi(s) = T_i\psi(s)/(T_e + T_i)$, while the effective barrier for electrons is $\psi(s) - e\varphi(s) = T_e\psi(s)/(T_e + T_i)$. The current density of ions passing over the barrier can now be calculated from the first moment of the ion distribution function evaluated at $s = s_m$, where ψ and φ have their maximum values ψ_m and φ_m:

$$J = J_0 \exp\{-[\psi_m/k(T_e + T_i)]\}. \quad (8)$$

Here, J_0 is the ion current density in the absence of rf. Substitution of $\psi_m = e^2 E_m{}^2/4m\omega^2$ into (8) gives a second criterion [after (4)] for effective rf plugging, viz.

$$E_m \gtrsim (2\omega/e)[mk(T_e + T_i)]^{1/2}, \quad (9)$$

where E_m is the peak value of $E_\parallel(s)$.

The conditions $v_0 \ll \omega L$ and $\tau_R \gg L/v_0$ were satisfied in these experiments, since for 10-eV electrons $v_0 = 1.3\times10^8$ cm/sec, while ω was 1.5×10^8 sec^{-1}, τ_R was 10 μsec, and L can be approximated by the antenna length, 9 cm. We believe that plasma plugging by the ponderomotive effect on electrons has been demonstrated because (1) the rf field strengths were too small for the ponderomotive effect on ions, but were in the range required for electrons; (2) the pattern of charge-collector responses corresponded to the rf field pattern of the Nagoya type-III antenna; and (3) the response of the East collector varied with rf power in accordance with the prediction of the theory. We have shown that, in rf plugging experiments, whenever $E_\parallel$ is a nonnegligible fraction of $E_\pm$, the ponderomotive effect on electrons must be considered for its role in plugging.

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[1]T. Watari, T. Hatori, R. Kumazawa, S. Hidekuma, T. Aoki, T. Kawamoto, M. Inutake, S. Hiroe, A. Nishizawa, K. Adati, T. Sato, T. Watanabe, H. Obayashi, and K. Takayama, Phys. Fluids 21, 2076 (1978).
[2]H. Motz and C. J. H. Watson, Adv. Electron. 23, 153 (1967).
[3]Y. Ohsawa, M. Inutake, T. Tajima, T. Hazuri, and T. Kamimura, Phys. Rev. Lett. 43, 1246 (1979).
[4]A. F. Haught, W. B. Ard, W. J. Fader, R. A. Jong, A. E. Mensing, D. H. Polk, R. G. Tomlinson, and J. T. Woo, in *Plasma Physics and Controlled Nuclear Fusion Research—1974* (International Atomic Energy Agency, Vienna, 1975), Vol. I, p. 391.
[5]H. Schamel, Phys. Rev. Lett. 42, 1339 (1979).

Omnigenous Equilibria for Charged Particles in a Magnetic Field

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Scalar pressure equilibria in which the divergence of the parallel current density vanishes are shown to be free of neoclassical transport (omnigenous), for any asymmetry, any plasma β, and in any collisionality regime. The explicit constraint that eliminates neoclassical effects in a long, thin mirror equilibrium in the low-β limit is derived, and some model fields considered.

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Omnigenous equilibria are defined by Hall and McNamara[1] as equilibria in which all charges on a given magnetic field line drift in the same flux surface. The drift velocity $\vec{v}_d$ is the sum of the magnetic, $\vec{v}_m$, plus electric drifts, $\vec{v}_e$.

Previous investigations[1,2] of omnigenous equilibria have attempted to determine the conditions under which flux surfaces (surfaces of constant α) and surfaces of constant $J \propto \oint dl\, v_\parallel$, the second adiabatic invariant, could be made coincident. However, the validity of J as an invariant requires a bounce average description so that only the bounce average location of charges can be treated. Consequently, a charge can have zero bounce average drift away from a flux surface, $\langle \vec{v}_d \cdot \nabla\alpha\rangle \propto \oint (dl/v_\parallel)(\vec{v}_d \cdot \nabla\alpha) = 0$, yet continue to have local departures of $\vec{v}_d \cdot \nabla\alpha$ from zero. The present treatment is most relevant for those species (i.e., ions in a tandem) not adequately described by bounce average drift kinetics.

Because of the importance of cross field or "radial" transport in various fusion devices, it is critical to know whether there exist three-dimensional, nonaxisymmetric equilibria which are *locally* omnigenous, that is, $\vec{v}_d \cdot \nabla\alpha = 0$ at each point. The distinction between $\langle \vec{v} \cdot \nabla\alpha \rangle$ and $\vec{v}_d \cdot \nabla\alpha$ is crucial, since local omnigenity implies a complete absence of neoclassical transport. The averaged condition, on the other hand, while relatively easy to satisfy,[1,3,4] allows neoclassical processes to occur whenever the transit time is too long to permit a bounce average description.

In order to understand locally omnigenous equilibria, recall that neoclassical transport in nonaxisymmetric devices is not automatically ambipolar.[5] The magnetic drift dependence on charge and mass results in charge separations that generate radial and azimuthal (or poloidal) variations in the electrostatic potential. The resulting azimuthal and radial $\vec{E}\times\vec{B}$ drifts, $\vec{v}_e$, locally adjust the electric field $\vec{E}$ to establish quasineutrality and stable ambipolar operation. Because the magnetic drift $\vec{v}_m$ that carries charges off flux surfaces is the drive, when $\vec{v}_m \cdot \nabla\alpha = 0$, then $\vec{v}_e \cdot \nabla\alpha = 0$ as well (assuming an $\vec{E}$ is not generated by other means). Consequently, locally omnigenous equilibria occur when $\vec{v}_m \cdot \nabla\alpha = 0$.

By employing only the pressure balance equation

$$c^{-1}\vec{J}\times\vec{B} = \nabla p = (\nabla\alpha)dp/d\alpha \tag{1}$$

and the Maxwell's equations $\nabla\cdot\vec{B} = 0$ and

$$\nabla\times\vec{B} = (4\pi/c)\vec{J}, \tag{2}$$

where $p = p(\alpha)$ is the pressure, $\vec{J}$ the current density, and $\vec{B}$ the magnetic field, it will be shown that the requirement of local omnigenity ($\vec{v}_m \cdot \nabla\alpha = 0$) is equivalent to

$$\nabla\cdot(J_\parallel\hat{n}) = 0, \tag{3}$$

with $\hat{n} = \vec{B}/B$, $B = |\vec{B}|$, and $J_\parallel = \hat{n}\cdot\vec{J}$. The flux surfaces α are taken to be defined by the surfaces of constant pressure p.

To prove the equivalence of $\vec{v}_m \cdot \nabla\alpha = 0$ and Eq. (3), it is first necessary to note that Eqs. (1) and (2) may be combined to show that

$$\nabla\alpha\cdot(\vec{B}\times\nabla\ln B) = \nabla\alpha\cdot[\vec{B}\times(\hat{n}\cdot\nabla\hat{n})]. \tag{4}$$

Consequently, Eq. (4), along with the magnetic drift

$$\vec{v}_m = \Omega^{-1}\hat{n}\times(\mu\nabla B + v_\parallel{}^2\hat{n}\cdot\nabla\hat{n}), \tag{5}$$

where Ω is the gyrofrequency, permits $\vec{v}_m\cdot\nabla\alpha$ to be written in the two equivalent forms,

$$\vec{v}_m\cdot\nabla\alpha = [(\mu B + v_\parallel{}^2)/\Omega B^3]\nabla\alpha\cdot[\vec{B}\times(\vec{B}\cdot\nabla\vec{B})] \tag{6}$$

and

$$\vec{v}_m\cdot\nabla\alpha = [(\mu B + v_\parallel{}^2)/\Omega B^2]\nabla\alpha\cdot(\vec{B}\times\nabla\vec{B}). \tag{7}$$

The quantities μ and $v_\parallel$ are the magnetic moment and parallel velocity component.

To complete the proof, Eq. (1) may be employed to obtain

$$\vec{J}_\perp \equiv \hat{n}\times(\vec{J}\times\hat{n}) = (c/B^2)\vec{B}\times\nabla p, \tag{8}$$

which may then be inserted in $\nabla\cdot\vec{J} = 0$ to yield

$$\nabla\cdot(J_\parallel\hat{n}) = -\nabla\cdot\vec{J}_\perp = (2c/B^3)\vec{B}\times\nabla p\cdot\nabla B. \tag{9}$$

Combining Eqs. (7) and (9) gives the desired result,

$$\vec{v}_m\cdot\nabla\alpha = [-B(\mu B + v_\parallel{}^2)/2c\Omega(dp/d\alpha)]\nabla\cdot(J_\parallel\hat{n}). \tag{10}$$

Therefore a scalar pressure equilibrium is locally omnigenous ($\vec{v}_m\cdot\nabla\alpha = 0$) if and only if its parallel (or perpendicular) current density has no divergence.

Because Eq. (10) is derived without demanding any symmetry of the magnetic field, one might anticipate that it is possible to design magnetic field configurations in which $\nabla\cdot(J_\parallel\hat{n})$ is zero or, at least, very small. In the following paragraphs one class of mirror devices in which transport can be of concern[4,6,7] is examined within the long-thin (or paraxial) approximation, $\epsilon \sim |\hat{n}\cdot\nabla| / |\hat{n}\times\nabla| \ll 1$. Only devices having quadrupole fields, rotated by 90° with respect to the axial midplane ($z = 0$), are considered.

Within the long-thin ordering, Eq. (6) requires less precise expressions for $\vec{B}$ than Eq. (7). In fact, the vacuum fields may be employed consistently[3] in Eq. (6). For a low-β plasma, $\beta_p \equiv 8\pi p/B^2 \sim \epsilon$, these fields in cylindrical (r,θ,z) variables may be written as[3,8]

$$\vec{B} = \vec{B}_0 + \vec{B}_1 = B_0(z)\hat{z} + \nabla_\perp\varphi_1, \tag{11a}$$

$$\alpha = \tfrac{1}{2}r^2[\bar{\alpha}(z) - \tilde{\alpha}(z)\cos 2\theta], \tag{11b}$$

$$\varphi_1 = \tfrac{1}{4}r^2 B_0[\bar{e}(z) + \tilde{e}(z)\cos 2\theta], \tag{11c}$$

where $|\vec{B}_1/B_0| \sim \epsilon \ll 1$. Demanding $\nabla\cdot\vec{B} = 0 = \vec{B}\cdot\nabla\alpha$ gives the following results (for the desired field

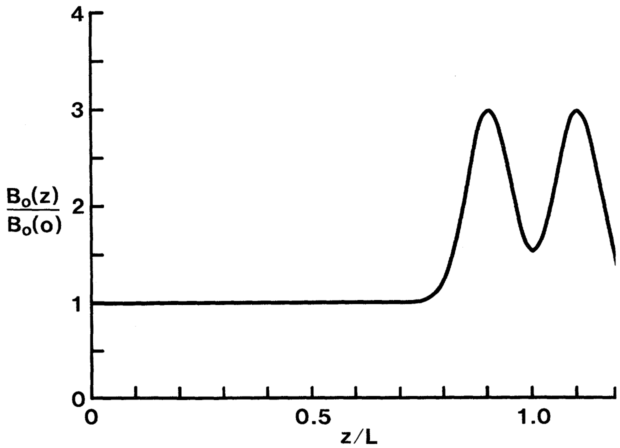

FIG. 1. Normalized model tandem field $B_0(z)/B_0(0)$ vs the normalized distance z/L along the magnetic axis. The length of the tandem, between the plug centers, is $2L$.

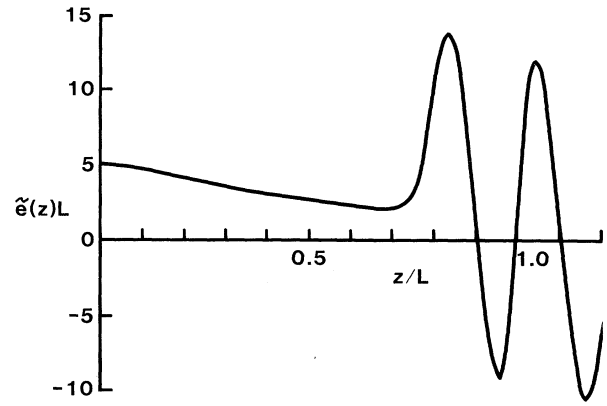

FIG. 2. The dimensionless fanning function $\tilde{e}(z)L$ vs z/L when $\tilde{e}(0)L = 5.0$, for the field of Fig. 1.

symmetry and within the long-thin ordering):

$$\bar{e} = -B_0'/B_0, \tag{12a}$$

$$\bar{\alpha} = B_0 \cosh f, \tag{12b}$$

$$\tilde{\alpha} = B_0 \sinh f, \tag{12c}$$

$$f = \int_0^z dz' \tilde{e}(z'), \tag{12d}$$

where prime denotes differentiation with respect to z. In general, Eqs. (11) and (12) are seen to permit an arbitrary specification of $B_0 = B_0(z)$ and $\tilde{e} = \tilde{e}(z)$, both even functions of z.

For a locally omnigenous equilibrium, however, the additional constraint $\vec{v}_m \cdot \nabla\alpha = 0$ must be satisfield. Inserting Eqs. (11) and (12) into Eq. (6) yields the local omnigenity requirement

$$\tilde{\alpha}\bar{e}' + \bar{\alpha}\tilde{e}' + \bar{\alpha}\bar{e}\tilde{e} + \tfrac{1}{2}\tilde{\alpha}(\bar{e}^2 + \tilde{e}^2) = 0 \tag{13}$$

or

$$\frac{d}{dz}\left[\frac{\tilde{e}(\cosh f)^{1/2}}{B_0}\right] = \left[\frac{B_0''}{B_0^2} - \frac{3(B_0')^2}{2B_0^3}\right]\frac{\sinh f}{(\cosh f)^{1/2}}. \tag{14}$$

Equations (13) and (14) are unchanged when a long-thin, low-β anisotropic plasma is considered.

Normally, for open-ended mirror devices one simply demands that $J_\parallel$ vanish at each end of the machine. With use of Eqs. (4) and (9) to write

$$\hat{n} \cdot \nabla(J_\parallel/B) = -(2c/B^5)(dp/d\alpha)\nabla\alpha \cdot [\vec{B} \times (\vec{B} \cdot \nabla\vec{B})],$$

integration from end to end along a field line gives the constraint[9]

$$\oint (dl/B^5)\nabla\alpha \cdot [\vec{B} \times (\vec{B} \cdot \nabla\vec{B})] = 0.$$

For 90°-rotated quadrupole fields, this constraint is automatically satisfied and such fields are sometimes said to possess omnigenous "symmetry,"[3,4] even though $\vec{v}_m \cdot \nabla\alpha \neq 0$ in general.

Equation (14) gives a unique solution for $\tilde{e}(z)$ for a specified $B_0(z)$ and a given $\tilde{e}(z=0) \neq 0$. This last restriction means that a tandem having a cylindrical center solenoid that is *strictly* circular cannot be made locally omnigenous.[10] Consequently, for locally omnigenous long-thin mirror equilibria only $B_0(z)$ *or* $\tilde{e}(z)$ may be specified.

In the simplest possible case $B_0(z)$ is constant; then Eq. (14) gives

$$\tilde{e} = \tilde{e}(0)/(\cosh f)^{1/2}. \tag{15}$$

In this case, $\tilde{e}$ simply drops monotonically to zero as a function of z.

As an example of a $B_0(z)$ that is of more practical interest, the model "tandem" field of Fig. 1 is inserted in Eq. (14). Numerical integration then yields the $\tilde{e}(z)$ shown in Fig. 2 for a specified $\tilde{e}(0)$. For this same $\tilde{e}(0)$, Figs. 3(a) and 3(b) are plots of the Cartesian ($x = r\cos\theta$, $y = r\sin\theta$) location of the field line,

$$x \propto B_0^{-1/2}\exp(f/2)$$

and

$$y \propto B_0^{-1/2}\exp(-f/2),$$

that passes through $x(0)$ and $y(0)$ at $z=0$. While physics and/or engineering considerations may make this particular field configuration unsatisfactory, Figs. 1–3 nonetheless illustrate the existence and potential importance of locally omnigenous equilibria.

One important physics issue is that of fluid sta-

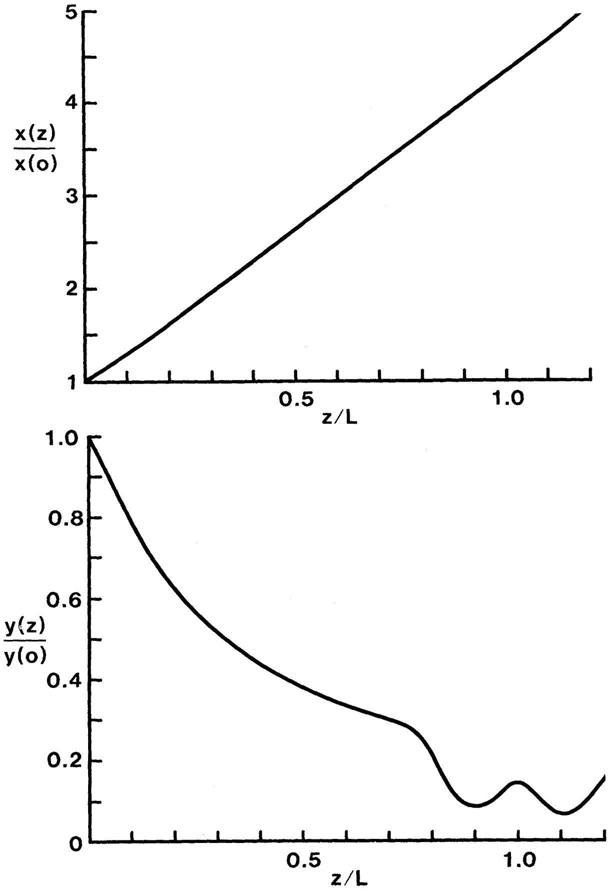

FIG. 3. The Cartesian locations of the field line normalized to the $z=0$ values, as a function of z/L for $\tilde{e}(0)L=5.0$ and the $B_0(z)$ of Fig. 1.

bility. It might be thought, for example, that locally omnigenous equilibria are interchange unstable.[11] However, within the usual eikonal *Ansatz*, the criterion for interchange stability (for a normal pressure profile) for these $\nabla\cdot(J_{\parallel}\hat{n})=0$ fields may be written as

$$I=\oint\frac{dz}{B_0^2\cosh f}\left[\tilde{e}^2+3\left(\frac{B_0'}{B_0}\right)^2-\frac{2B_0''}{B_0}\right]>0. \qquad (16)$$

For the case considered in Figs. 1–3, I is indeed positive; the integrand of I is plotted in Fig. 4. In general, interchange stability is assured for a large enough $\tilde{e}(0)$. Increasing the distance between the plugs and/or the spread of the plug fields, and/or decreasing the plug length, enhances stability.

While only locally omnigenous mirror equilibria have been considered here, it seems likely that such equilibria are possible, and perhaps better suited, to other devices. In particular, for closed

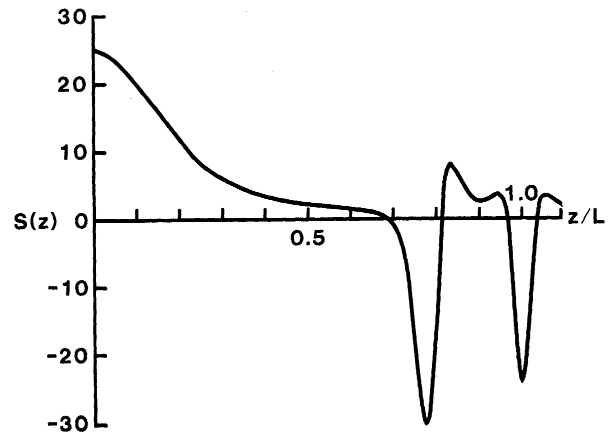

FIG. 4. The stability "integrand" $S(z)$, where S is defined as $B_0^2(0)$ times the integrand of Eq. (16), vs z/L for $\tilde{e}(0)L=5.0$ and the B_0 of Fig. 1. The stability integral $\oint dz\, S(z)=B_0^2(Q)I=7.8$ when S is integrated between $z/L=\pm 1.1$.

geometries (such as the Elmo bumpy torus, and perhaps stellarators) in which cross field transport is the only loss mechanism, locally omnigenous (or nearly locally omnigenous) equilibria offer the hope of dramatically improved confinement.

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[1]L. S. Hall and B. McNamara, Phys. Fluids 18, 552 (1975).

[2]M. E. Rensink, T. K. Fowler, R. P. Freis, J. Killeen, A. A. Mirin, R. W. Moir, L. D. Pearlstein, R. F. Post, C. G. Tull, L. S. Hall, B. McNamara, J. K. Boyd, C. A. Finan, D. Fuss, and C. A. Wilgus, in *Plasma Physics and Controlled Nuclear Fusion Research—1974* (International Atomic Energy Agency, Vienna, 1974), Vol. I, p. 311.

[3]"Status of Mirror Fusion Research 1980," edited by B. I. Cohen, Lawrence Livermore National Laboratory Report No. UCAR-10049-80-Rev. 1, 1980 (unpublished), (also available from National Technical Information Service, Virginia), pp. 2-22–2-25.

[4]D. E. Baldwin, Lawrence Livermore Laboratory Internal Report No. UCID-17926, 1978 (unpublished).

[5]R. D. Hazeltine and P. J. Catto, Phys. Fluids (to be

published).

[6]D. D. Ryutov and G. V. Stupakov, Pis'ma Zh. Eksp. Teor. Fiz. 26, 186 (1977) [JETP Lett. 26, 174 (1977)], and Fiz. Plasmy 4, 501 (1978) [Sov. J. Plasma Phys. 4, 278 (1978)], and Dokl. Akad. Nauk SSSR 240, 1086 (1978) [Sov. Phys. Dokl. 23, 412 (1978)].

[7]R. H. Cohen, Nucl. Fusion 19, 1579 (1979).

[8]H. P. Furth and M. N. Rosenbluth, Phys. Fluids 7, 764 (1964); J. G. Cordey and C. J. H. Watson, in *Proceedings of the British Nuclear Energy Society Nuclear Fusion Reactor Conference*, edited by J. L. Hall and J. H. C. Maple (British Nuclear Energy Society, Culham, 1970), p. 122.

[9]W. A. Newcomb, Phys. Fluids 2, 362 (1959); J. B. Taylor, Phys. Fluids 6, 1529 (1963).

[10]G. V. Stupakov, to be published (translation available as Lawrence Livermore Laboratory Translation Report No. LLL-Ref-02688, June 1979).

[11]M. N. Rosenbluth and C. L. Longmire, Ann. Phys. (N.Y.) 1, 120 (1957).

Kosterlitz-Thouless Transition in the Two-Dimensional Plane Rotator and Coulomb Gas

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We describe a rigorous argument establishing the Berezinski-Kosterlitz-Thouless transition in a class of two-dimensional models including the plane rotator and the Coulomb gas. The main idea is to rewrite correlations in the Coulomb gas as superpositions of correlations in gases of *neutral molecules* of variable size and small effective activity.

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In recent years a number of authors have given fairly convincing arguments for the existence of a phase transition and a line of critical points in a class of two-dimensional (2D) models, including the rotator model, the related Villain model, and the lattice Coulomb gas. (See Refs. 1 and 2 and references given therein.) All models are known to have a high-T (T=temperature) phase with exponentially decaying correlations. The presumed transition is one from the high-T to a low-T phase characterized by a power-law falloff of correlations and scaling, so that all temperatures T below some positive T_c are critical points. Since the rotator model, and the Villain model, have a continuous, global U(1) symmetry, this transition is not accompanied by symmetry breaking, and there is no spontaneous magnetization—a well-known consequence of Mermin's theorem.[3]

Perhaps the best arguments for the existence of the transition described above are based on analyzing the low-T behavior of the 2D Coulomb gas. Consider a positive and a negative charge separated by some distance l. They can be viewed as forming a neutral dipole whose Boltzmann factor is $\propto \exp[-(\beta/2\pi)\ln(l+1)]$, where β is the inverse temperature. Moreover, when the gas is very dilute, dipoles are the dominant configurations, and the exponential of the mean entropy of a dipole of length l is $\propto l^3$. Thus, the probability for such a dipole to be present is $\propto l^3 \exp[-(\beta/2\pi)\ln(l+1)]$ which is summable in l when $\beta > 8\pi$. Therefore, the 2D Coulomb gas at large β is expected to behave like a dipole gas. It is known that correlations in a dipole gas have a power-law falloff.[4,5] In this paper we sketch a proof of existence of the transition described above—henceforth called the Kosterlitz-Thouless (KT) transition—for the simplest model, namely a dilute 2D Coulomb gas, which is inspired by the above heuristic argument. Details of our proof, as well as extensions to other 2D models and higher-dimensional, Abelian lattice gauge theories, will appear elsewhere.[6]

We now describe some of our main results concerning 2D models: For sufficiently low temperatures, (i) the spin-spin correlation in the plane rotator and the Villain model has a power-law falloff and (ii) the Coulomb gas does not screen;

