The lighter perturbers He and Ne have rather small values of a_1 indicating the importance of the repulsive parts of the potentials for these systems.

Our measurements confirm the prediction that the collision-broadened line shape manifests a significant dispersion component in the core region in addition to the well-known Lorentzian. The dispersion component is due to the finite duration of collisions (T_d) and produces an asymmetric line shape with a red-blue asymmetry of order $|\omega-\omega_0|T_d$. The dispersion component accurately accounts for the difference between observed line shapes and a Lorentzian. Hence the core-region line shape can be characterized by three parameters—the Lorentzian width and shift, and the asymmetry parameter. The asymmetry parameters have been determined with less than 10% error, but it has not been possible to check corresponding theoretical predictions to this precision. The scalar theories[5,6] do not contain prescriptions for including the three excited-state potentials dissociating to the Na $3p$ fine-structure levels, hence accurate theoretical calculations have not been possible even where the potentials are accurately known. Calculations based on more sophisticated theories[15] would be most welcome, and would have the advantage of being immediately testable at the (5–10)% level.

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Subcritical Transition to Turbulence in Plane Channel Flows

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A linear three-dimensional instability mechanism is presented that predicts Reynolds numbers for transition to turbulence in plane channel flows in good agreement with experiment.

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While experiments[1] show that incompressible plane Poiseuille and plane couette flow may undergo transition to turbulence at Reynolds numbers $\boldsymbol{R}$ of order 1000, linear stability analysis of these plane parallel flows gives critical Reynolds numbers of 5772 for plane Poiseuille flow[2] and ∞ for plane couette flow.[3] This discrepancy between theory and experiment suggests that the mechanism of transition is not properly represented by parallel-flow linear stability analysis. In this Letter, we present a new linear three-dimensional mechanism that predicts transition at Reynolds numbers in good agreement with experiment for both plane Poiseuille and plane couette flows. Here we present the theory applied to plane Poiseuille flow, defined as flow between fixed parallel plates that is driven by a pressure gradient.

We begin by studying two-dimensional traveling-wave solutions to the Navier-Stokes equations

$$\vec{v}(x, z, t) = \vec{F}(x - ct, z), \qquad (1)$$

where c is a real wave speed, x is the downstream coordinate and z is the coordinate perpendicular to the channel walls at $z = \pm 1$. No-slip boundary conditions are applied at the walls and $2\pi/\alpha$ periodicity in x is assumed. For all R, one solution is the laminar flow $(1 - z^2)$, where $R = 1/\nu$ and ν is the kinematic viscosity. For $R \gtrsim 2900$, up to two other solutions (neglecting an arbitrary phase) may exist for any given α.[4] The locus of points in (E, R, α) space for which these solutions exist is called the neutral surface. Here E is the energy of the flow relative to that of the laminar flow. A slice of the neutral surface for given subcritical Reynolds number $(2900 \lesssim R < 5772)$ is shown in Fig. 1.

If a one-dimensional phase-space representation were appropriate to describe the behavior of flows off the neutral surface, E would evolve according to

$$dE/dt = f(E). \qquad (2)$$

Typically the critical points of (2) are alternately stable and unstable, and so the lower-branch (LB) solutions on the subcritical neutral surface plotted in Fig. 1 are unstable while the upper-branch (UB) solutions are stable.

While these stability predictions are correct, the evolution of two-dimensional flows is not restricted to a one-dimensional phase space. Projections of numerical solutions[5] of the two-dimensional Navier-Stokes equations on the two-dimensional phase space $[(E_1)^{1/2}, (E_2)^{1/2}]$ are plotted in Fig. 2. Here E_k is the kinetic energy in that part of the flow that depends on x like $e^{ik\alpha x}$. Orbits of solutions with initially large energies do not follow simple curves. The time-dependent evolution of two-dimensional flows evidently requires a multi- (likely infinite-) dimensional phase space. Thus, Landau-Stuart-Watson[6] nonlinear stability theory, which gives evolution equations of the form (2), cannot be valid away from the neutral surface.[7]

Several other features of Fig. 2 are noteworthy. First, the two orbits in the lower left-hand corner illustrate the existence of a threshold energy (near that of the LB solution) below which disturbances decay. Second, solutions with energies less than that of the UB solution (indicated by the point marked "steady solution" in Fig. 2) can overshoot the UB energy by factors of 4 or more. Third, and most importantly, typical solutions quickly evolve to a state within a band of quasiequilibria and, then, only very slowly approach the steady UB solution. The time scale for initial adjustment to a quasiequilibrium state is of order the eddy circulation time $1/\sqrt{E}$ (i.e.,

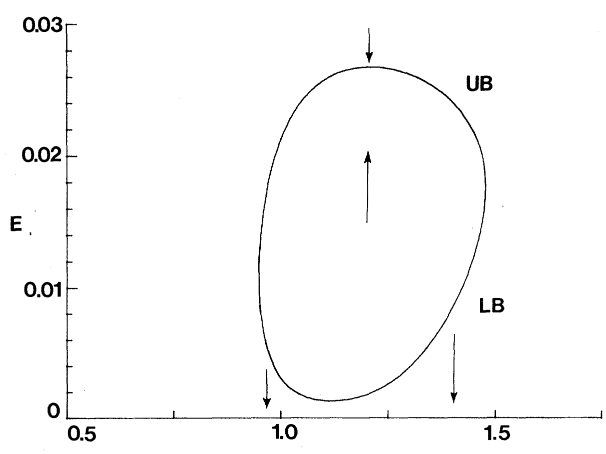

FIG. 1. A subcritical (E, α) slice of the neutral surface for plane Poiseuille flow at $R = 4000$. The stability of solutions is indicated by the arrows. The behavior shown in this plot is typical for $2900 \lesssim R < 5772$.

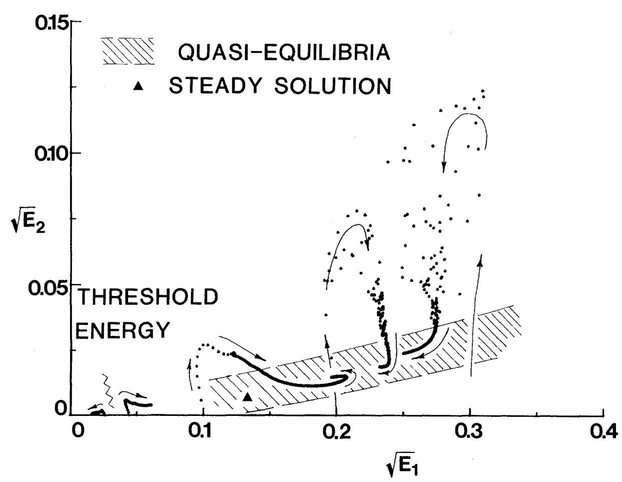

FIG. 2. A phase portrait of disturbances to laminar plane Poiseuille flow in $[(E_1)^{1/2}, (E_2)^{1/2}]$ space at $R = 4000$, $\alpha = 1.25$. The dots, equally spaced by 1.25 in time, indicate the evolution of perturbations from initial conditions proportional to the least stable Orr-Sommerfeld eigenfunction at this (α, R). Note the existence of a band of quasiequilibria.

of order 10), while the time scale for approach to the equilibrium state is of order the diffusion time $1/\nu$ (i.e., of order 1000–10 000). In the quasiequilibria, the spanwise vorticity must be nearly constant on streamlines,[8] so that equilibrium is achieved by diffusion of vorticity. In fact, vorticity can vary by at most $O(\nu)$ along interior streamlines of the equilibrium flows. Nearby flows must have the same property implying the existence of quasiequilibria evolving only on a diffusive time scale.

The quasiequilibria are the basis of our transition mechanism in plane Poiseuille flow as direct numerical solution of the Navier-Stokes equations[9] shows that they are strongly unstable to infinitesimal three-dimensional disturbances. In Fig. 3, we plot the evolution of (initially small) three-dimensional disturbances superposed on finite-amplitude two-dimensional motions. Evidently, the three-dimensional disturbances quickly achieve a form that grows exponentially in time for $R \gtrsim 1000$. The growth rate of the three-dimensional disturbances is rapid with their amplitude increasing by a factor of about 10 in a time of 10. This short time scale for subcritical three-dimensional growth should be contrasted with the long time scale of order 1000 for evolution of supercritical Orr-Sommerfeld modes.[2]

There is strong evidence that this instability is a physically relevant one in that it is fairly insensitive to initial conditions and has small threshold energies. It is necessary to distinguish here between this instability and the ensuing transition to turbulence. If the two-dimensional flow persists sufficiently long for the three-dimensional perturbations to attain a finite amplitude, direct numerical simulation has shown[9] that the resulting three-dimensional flow quickly develops a turbulent character with strongly nonperiodic behavior. Thus to "predict" transition one must know the initial two-dimensional and three-dimensional energies as well as their respective time scales. For instance, the most dangerous three-dimensional instability for given two-dimensional energy is not necessarily the most likely to force transition if the two-dimensional state is outside the band (in wave number) of quasiequilibria. It is possible to use our methods to construct a neutral surface for transition in any given (presumably large) parameter space. However, we confine attention here to demonstrating that our mechanism predicts transitional Reynolds numbers in accordance with experiment.

The exponential growth illustrated in Fig. 3 suggests that a linear instability mechanism is

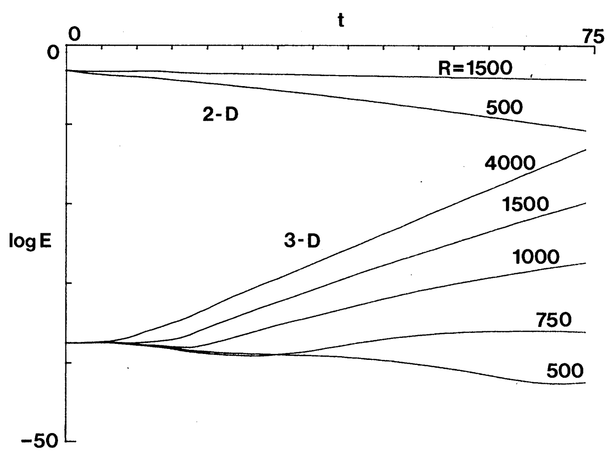

FIG. 3. A plot of the growth of three-dimensional perturbations on finite-amplitude two-dimensional states in plane Poiseuille flow at $(\alpha, \beta) = (1.32, 1.32)$. Here E_{2D} is the total energy (relative to the laminar flow) in wave numbers of the form $(n\alpha, 0)$, while E_{3D} is the total energy in wave numbers $(n\alpha, \beta)$. For $R \gtrsim 1000$ we obtain growth and for $R = 500$ decay. The growth rate of the three-dimensional disturbance *amplitude* at $R = 4000$ is about 0.18 ($\sim\sqrt{E_{2D}}$) and depends only weakly on R for larger R. The initial conditions are superpositions of the laminar flow, a [large ($E_{2D} = 0.04$)] two-dimensional Orr-Sommerfeld mode with wave vector $(\alpha, 0)$ and a [very small ($E_{3D} = 10^{-16}$)] three-dimensional transverse Orr-Sommerfeld mode with wave vector $(0, \beta)$.

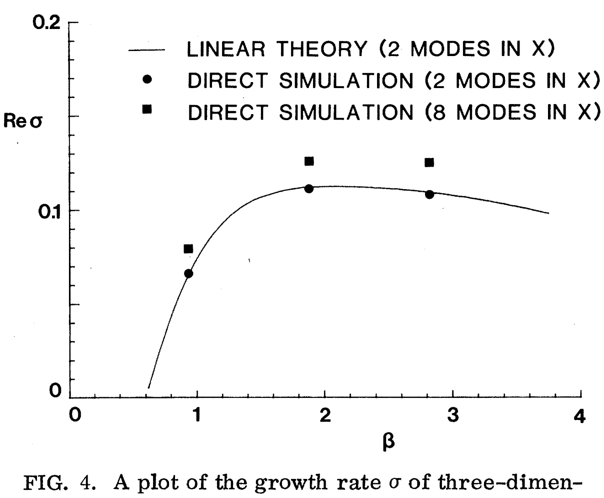

FIG. 4. A plot of the growth rate σ of three-dimensional perturbations as a function of β at $R = 4000$, $\alpha = 1.25$. Note the good agreement between the linear calculation and the two-mode direct simulations. Increasing the number of retained modes in x increases the growth rates. However, the error in the two-mode model is not large.

involved.[10] Assuming a flow of the form

$$\vec{v}(\vec{x}, t) = \vec{F}(x - ct, z) + \epsilon \mathrm{Re}[\vec{G}(x - ct, z)\exp(\sigma t + i\beta y)], \quad (3)$$

substituting into the Navier-Stokes equations, and linearizing with respect to ϵ, a linear eigenvalue problem for σ results. The Galilean transformation to a reference frame moving with the phase speed c eliminates time-dependent coefficients, so that the problem is separable in t. The resulting eigenvalue problem has been solved numerically with use of Chebyshev polynomial expansions in z and highly truncated Fourier series expansions in x for $\vec{F}$ and $\vec{G}$. In Fig. 4, we plot the maximum growth rate,[11] Re(σ), vs the spanwise wave number β for $R = 4000$, $\alpha = 1.25$. The results of direct numerical simulations (cf. Fig. 3) are also plotted in Fig. 4. Evidently, the large growth rates observed in the direct numerical simulations can be explained by this linear eigenvalue problem.

Note that the linear theory presented above can be extended to Reynolds numbers below 2900 by freezing the quasiequilibria which evolve very slowly compared to the rapid exponential growth of the three-dimensional perturbations. For $R \gtrsim 1000$, the quasiequilibria decay sufficiently slowly that three-dimensional perturbations can grow, overwhelm the two-dimensional flow, and break down to turbulence.

The rapid growth rates described above are due to the *combined* action of vortex stretching by the nearly inviscid two-dimensional steady motion $\vec{F}$ and tilting of the vortex lines of $\vec{F}$ by the perturbation $\vec{G}$. By itself, vortex stretching by $\vec{F}$ cannot give rapid exponential growth rates because of the two-dimensional antidynamo theorem.[12] Detailed flow visualizations of the instabilities described here will be given elsewhere. It will be shown that three-dimensional perturbations grow on a time scale of order $1/\sqrt{E_{2\mathrm{D}}}$, which must be shorter than the decay time of the two-dimensional motion for the instability to be effective. The sharp cutoff in growth rate Re(σ) for small β observed in Fig. 4 reflects a threshold of streamwise vorticity for stretching to persist.

Direct numerical simulations[9] of transition in plane couette flow show that while there is no evidence that equilibria of the form (1) exist, the three-dimensional instability process outlined above is still effective down to Reynolds numbers of order 1000. While there are no quasiequilibria in plane couette flow that evolve on purely diffusive time scales, the decay rates of finite-amplitude two-dimensional disturbances are still several eddy circulation times. This implies that the threshold three-dimensional energies in plane couette flow are somewhat larger than in plane Poiseuille flow. However, the resulting instability is at least as strong and turbulence quickly ensues.

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Dynamics of Cavitons at Critical Density

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The dynamics of envelope solitons accompanied by density depressions (cavitons) is analyzed with use of the driven Zakharov equations for inhomogeneous plasmas. The new contributions due to ion inertia as well as the novel phenomenon of amplitude-width-symmetry breaking are discussed. The results are applied to resonance absorption processes in laser-produced plasmas.

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In an inhomogeneous plasma, conversion of electromagnetic waves into electrostatic waves enhances the field strengths of the latter by several orders of magnitude in the vicinity of the resonant layer, where the incident frequency matches the local plasma frequency. The amplitude swelling is due to the reduction of the group velocity from the velocity of light c to the electron thermal velocity $v_{\mathrm{th}e}$. The enhancement factor is approximately $(c/v_{\mathrm{th}e})^{1/2}$, and thus nonlinear effects[1] play an important role.

The importance of radiation-pressure effects in laser-plasma interaction has been demonstrated by particle simulations.[2] According to these results, a variety of processes occur at critical density. Some of the most important are resonant heating of electrons and generation of suprathermal particles, strong steepening of density profile, etc. In the past, simplified analytical models[2,3] have been proposed to explain these basic physical phenomena. In this Letter, we focus on one of them, i.e., profile steepening. A similar problem was studied in Ref. 3, where profile steepening because of cavity formation could be predicted. However, the calculations were based on the driven nonlinear Schrödinger equation ignoring ion inertia. We believe that ion inertia effects can become important. For example, in the static approximation ($\partial n/\partial t \simeq 0$), it was found that a soliton would be strongly accelerated down the density gradient reaching even electron thermal velocities. We expect that because of coupling with ions, the acceleration should be much smaller than predicted by the cubic nonlinear Schrödinger equation. Then the soliton stays for a longer time in the resonant region. Since position, velocity, phase detuning, etc., are coupled in a highly nontrivial manner, the problem of profile steepening at the critical density in laser-created plasmas via soliton formation has to be reconsidered.

The formulation presented here consists of describing the evolution of the electric field E through the nonlinear Schrödinger equation in which the density modification n is obtained from the ion-acoustic wave equation with the effects of the ponderomotive force included self-consistently. Thus, in the one-dimensional electrostatic approximation and for small driving fields E_d, the basic equations are

$$i\epsilon\, \partial E/\partial t + \partial^2 E/\partial x^2 - (\alpha x + n)E = E_d\,, \tag{1}$$

$$\partial n/\partial t = -\partial u/\partial x\,, \tag{2}$$

$$\partial u/\partial t = -(\partial/\partial x)(n + EE^*)\,. \tag{3}$$

Here, the following units are used: time, $\sqrt{3}/\omega_{pi}$, where ω_{pi} is the ion plasma frequency; length, $\sqrt{3}\,\lambda_e$, where λ_e is the electron Debye radius; potential, T_e/e; density, N_0; electric field, $(4\pi N_0 T_e)^{1/2}$; and velocity, $c_s = (T_e/m_i)^{1/2}$. Because of this normalization the parameter $\epsilon = 2(m_e/3m_i)^{1/2}$ appears in Eq. (1). In actual experiments

