[1]Ch. Gruber and Ph. Martin, to be published.
[2]Ph. Choquard, Ch. Gruber, and Ph. Martin, to be published.
[3]G. L. Uhlenbeck, in *Fundamental Problems in Statistical Mechanics*, edited by E. G. D. Cohen (North-Holland, Amsterdam, 1968), p. 17.
[4]M. S. Wertheim, J. Chem. Phys. 65, 2377 (1976).
[5]Ch. Gruber, Ch. Lugrin, and Ph. Martin, J. Statist. Phys. 22, 193 (1980).
[6]Ch. Gruber, Ph. Martin, and J. L. Lebowitz, "Sum Rules in Nonhomogeneous Charged Fluids" (to be published).
[7]H. Kunz, Ann. Phys. (N. Y.) 85, 303 (1974).

Critical Phenomena on Fractal Lattices

Yuval Gefen
Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel

and

Benoit B. Mandelbrot[(a)]
Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

and

Amnon Aharony[(b)]
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 12 May 1980)

Renormalization-group techniques are applied to Ising-model spins placed on the sites of several self-similar fractal lattices. The resulting critical properties are shown to vary with the (noninteger) fractal dimensionality D, but also with several topological factors: ramification, connectivity, lacunarity, etc. For any $D \geq 1$, there exist systems with both $T_c = 0$, and $T_c > 0$; hence a lower critical dimensionality is not defined. The nonvanishing values of T_c and the critical exponents depend on all these factors.

PACS numbers: 05.70.Jk, 05.50.+q, 64.60.Fr, 75.40.Dy

Noninteger dimensionalities have recently entered physics from at least two separate directions: continuous ϵ expansions near an integer d in the theory of critical phenomena,[1] and fractals.[2] The ϵ expansions involve formal analytic continuations of momentum integrals, e.g., $\int d^d q \to \int q^{d-1} dq$,[1] or of recursion relations constructed for d-dimensional hypercubic lattices.[3] In these cases, *translational invariance* is assumed (without actual implementation). The resulting general belief is that for systems of given symmetry of the order parameter and interaction range, the critical properties depend *solely* on the dimensionality d.[4] In particular, all Ising models with short-range interactions and given $d \geq 1$ are believed to exhibit identical critical properties, with the transition temperature T_c decreasing to zero at the lower critical dimensionality $d_l = 1$. Unfortunately, because of the *purely formal* character of the analytic continuations, these beliefs cannot be tested.

By contrast, fractals[2] are *fully explicitly described* geometric shapes, which one may view as "hybrids" between standard (integer d) shapes such as lines or planes. A fractal's description involves *several* factors that can vary largely independently of one another: the fractal dimensionality D,[5] which is usually not an integer, the topological dimensionality,[6] the order of ramification,[7] the connectivity Q,[8] the lacunarity,[9] etc. Note that fractal lattices are scale invariant, but not translationally invariant.

The present Letter reports on the first systematic study of *critical phenomena on fractals*, namely in *spin systems carried by self-similar fractal lattices*.[10] Note that unlike the formal continuations, fractals are themselves *implemented* in real physical systems, e.g., percolation clusters.[2,11] The picture emerging from our application of the renormalization-group techniques to suitably varied fractals is *more complex and subtle* than the present conventional view. *A lower critical dimensionality is not defined.* In fact, the progression between successive integer dimensionalities can be performed in diverse ways, involving very different critical points. As a general rule, Ising systems with given D have $T_c = 0$ if the minimum order or ramification, $R_{\min}$,[7] is finite, and $T_c > 0$ if $R_{\min}$ is infinite.

More specifically, (A) we analyze the Ising model on fractals akin to the "Sierpiński carpet,"[2] which have an infinite order of ramification and adjustable $1 < D < 2$. We show *exactly* that the fixed point at $T=0$ is stable, from which we infer that $T_c > 0$. We then use an extension of the Migdal-Kadanoff bond-moving renormalization-group approach[3] to calculate approximate values for T_c and for the correlation-length critical exponent ν [defined via $\xi \sim (T-T_c)^{-\nu}$]. *Both depend not only on D, but also on the system's connectivity*[8] *and lacunarity.*[9] (B) We solve *exactly* a number of Ising systems with finite $R_{\rm min}$. For these, one writes $\xi \sim t^{-\nu}$, with $t = \exp(-2K) = \exp(-2J/k_BT)$, where J is the nearest-neighbor exchange. We find that ν depends on $R_{\rm min}$ and on the ramification's homogeneity. When $R = R_{\rm min} = 2$ at *all* points (no branching), we find $\nu = 1/D$. Otherwise, ν increases with the extent of branching, and $\nu > 1/D$. (C) In comparing different examples with the same D, we find that ν increases with increasing $R_{\rm min}$, diverging to infinity at some $R_{\rm min} = R_c < \infty$. At R_c, one has[10] $\xi \sim \exp(A/t^2) \sim \exp[A\exp(4K)]$.

We hope that our results will stimulate a discussion of the comparative importance of the diverse fractal and topological factors affecting critical phenomena. In particular, we expect that our results will lead to a more precise description of the thermal properties of spins on *percolating clusters*[11] in terms of these factors.

Generally, we start with a "microscopic" lattice system, with a finite nearest-neighbor distance. On length scales of interest, which are much larger than this basic distance, the system looks self-similar. Ising spins are placed on its sites, and some assumptions are made on their (nearest-neighbor) interactions. We then increase the basic length scale by iterating the renormalization-group transformation. We studied the following examples:

Nonbranching Koch curves.—We find $T_c = 0$, e.g., for the system shown in Fig. 1(a), which has[5] $D = \ln 4/\ln 3 \simeq 1.26$ ($b=3$ and $N=4$) and $R=2$ at every point.[7] We create a physical model by placing "spins" on the lattice sites, and allowing nearest-neighbor interactions not only between $\langle ab\rangle$, $\langle bc\rangle$, $\langle cd\rangle$, and $\langle de\rangle$, but also between $\langle bd\rangle$. The bond $\langle bd\rangle$ is not a part of the geometrical curve, and is not dressed with additional bonds on smaller length scales [unlike Fig. 1(b); see below]. Next we define a dedecoration renormalization group,[10] by tracing over spins b, c, and d, and adding a new interaction between $\langle af\rangle$. For

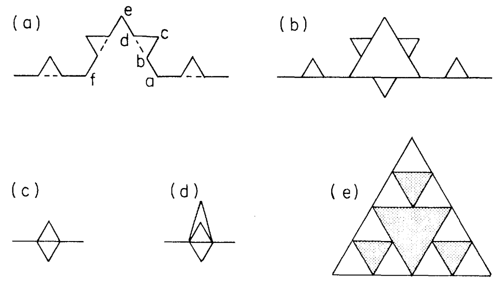

FIG. 1. (a) Two construction stages of the original (nonbranching) Koch curve. The broken lines represent spin interactions which are not part of the self-similar structure. (b)–(d) Various nonhomogeneous (branching) Koch curves. (e) Two stages of the Sierpiński gasket, constructed by a successive elimination of the shaded areas.

Ising spins, the exact recursion relation becomes $x' = x^4$, with $x = \tau^3(1+\tau)/(1+\tau^3)$ and $\tau = \tanh K$. Indeed, the only fixed points are at $T=0$ ($K=\infty$) or at $T=\infty$ ($K=0$). Linearization near $T=0$, writing $t' = b^y t$ [with $t = \exp(-2K)$], yields $1/\nu = y = \ln 4/\ln 3 = D$. The recursion relation $x' = x^N$, and the result $y = \ln N/\ln b$, hold for all nonbranching Koch curves with $R=2$. The familiar one-dimensional Ising model enters as the special case $y = D = 1$.[10]

Branching Koch curves.—The curves of Figs. 1(b), 1(c), and 1(d) are *nonhomogeneous*, in the sense that the order of ramification is $R=2$ at some points and larger than 2 at other (branching) points. Figs. 1(b), 1(c), and 1(d) differ by their dimensionality ($D = \ln 5/\ln 3$, $\ln 7/\ln 3$ and 2, respectively),[12] but they *all* yield the *same* $y = \ln 2/\ln 3 \simeq 0.63$. The reason for this is that the "density" of branching points is the same for all these curves.

Since we could replace t by some power of itself, the relation $\xi \sim t^{-\nu}$ is ambiguous. Moreover, distances on the D-dimensional Koch curves might be measured in non-Cartesian units, by replacing ξ by $\tilde{\xi} = \xi^D$. The result for the nonbranching Koch curve then becomes $\tilde{\xi} \sim e^{-2K}$, as expected for a "one-dimensional" Ising model. The surprising result is that this no longer holds for the branching case; in that case it seems that adding next—nearest-neighbor interactions in a self-similar way *does affect* the critical behavior, unlike the usual (nonself-similar) cases.

Sierpiński gasket.[2]—For the lattice of Fig. 1(e),[13] $D = \ln 3/\ln 2 = 1.585$ and $R = 3$ or 4 ("quasi-homogeneity").[7] The recursion relation here is

$$e^{4K'} = (e^{12K} + 3e^{4K} + 4)/(e^{8K} + 4e^{4K} + 3), \qquad (1)$$

reducing near $T=0$ to $(t^2)' = t^2 + 4t^4 + \ldots$. Hence

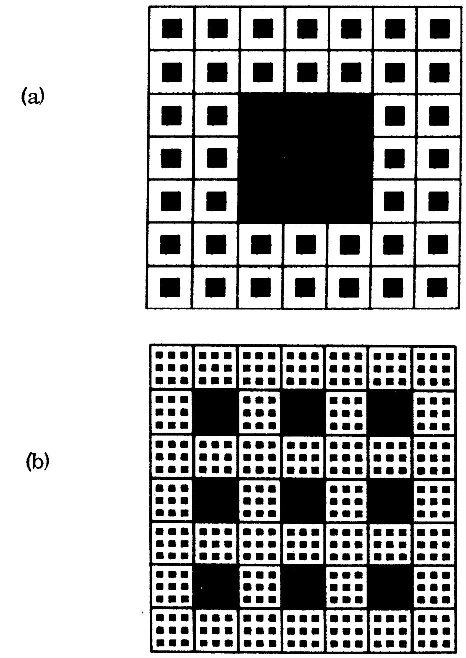

FIG. 2. Two stages of the Sierpiński carpets with $R=\infty$, $b=7$, and $l=3$. (a) Large lacunarity. (b) Small lacunarity.

$y=0$, $\nu=\infty$ and $\xi \propto \exp[4\exp(4K)]$.[10] Note that a nonbranching Koch curve with $N=3$, $b=2$ also has $D=\ln 3/\ln 2$, but yields $y=D$. Hence, ν is larger for the larger R and the larger density of branching points.

We have constructed various fractal curves with finite $R_{\min}$; *all* yield $T_c=0$.[14] This result is consistent with the standard inequalities or entropy arguments.[15]

Sierpiński carpets.—In the lattices of Fig. 2, $R=\infty$. They are constructed by subdividing a square into b^2 subsquares, then cutting out l^2 of these subsquares. Thus, $D=\ln(b^2-l^2)/\ln b$ [2,5] and $Q=\ln(b-l)/\ln b$.[8] D can range from nearly 1 [if $b \gg (b-l)$] to nearly 2 (if $b \gg l$). We first show exactly that $T_c>0$ for *all* these systems. At low T, the partition function is $Z=\exp(-E_0)[1+g_1e^{-6K}+O(e^{-8K})]$, where E_0 is the ground-state energy (in units of k_BT) and $6K$ represents the lowest excited state, in which one spin near a boundary of a cutout (shaded area) is flopped. A majority-rule renormalization group then yields *exactly*[14,16] $Z=Z'=Ae^{-E_0'}[1+g_1'e^{-mK}+\text{higher orders}]$, with $m>6$. Therefore, the exact *asymptotic* result near $T=0$ is $K'=(m/6)K>K$, and the fixed point $T=0$ is stable. Since the "paramagnetic" fixed point, at $T=\infty$, is also *always* stable, experience from analogous cases leads us to conclude that there exists at least one unstable fixed point at a *finite* value of T; hence $T_c>0$. Additional quantitative results at $T>0$ are obtained by approximate generalized Migdal-Kadanoff recursion relations. It is now necessary to distinguish between nearest-neighbor bonds on the boundary of a cutout, K_w, and internal bonds, K. Moving all the bonds within a dedecorated square to its perimeter and then decimating[3] [taking the eliminated subsquares to be in the center, Fig. 2(a)], we find

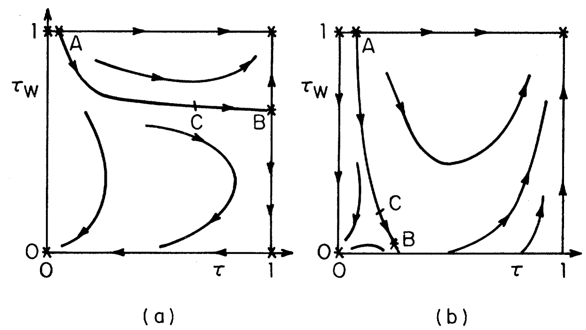

FIG. 3. Flow diagrams for Sierpiński carpets with $b=7$ and (a) $l=5$, (b) $l=3$. $\tau=\tanh K$, $\tau_w=\tanh K_w$. C is the critical point for $K=K_w$.

$$\tanh K' = \tanh^l[(b-l-1)K+2K_w]\tanh^{b-l}(bK),$$

$$\tanh K'_w = \tanh^l[\tfrac{1}{2}(b-l-2)K+2K_w] \times \tanh^{b-l}[\tfrac{1}{2}(b-1)K+K_w]. \quad (2)$$

(When $l=0$, we fall back to the known $d=2$ results.)[3] Combining analytic and numerical calculations, Eqs. (2) yield flow diagrams exemplified by Fig. 3. Fig. 3(a) relates to $b=7$, $l=5$ ($D\simeq 1.63$, $Q\simeq 0.36$). When $K_w=0$, the flows go from $K=\infty$ to $K=0$, implying $T_c=0$. The reason is that when $b-l=2$, setting $K_w=0$ lowers the order of ramification to a finite value, i.e., $R=2$ or 3 (without changing D!). The exponent y near $K=\infty$ ($K_w=0$) is $y=\ln l/\ln b$. The limit $K_w=\infty$ increases the connectivity, and yields a finite T_c characterized by the unstable fixed point A (Fig. 3).[17] Finite values of K_w yield criticality on the line ACB, flowing to the fixed point B, with $y(B)\simeq 0.34$. This line crosses the diagonal $K=K_w$ at C, yielding T_c via $\tanh K_c \simeq 0.67$. Similar flow diagrams arise for other values of b and $l=b-2$. A notable exception is $b=3$, $l=1$ ($D\simeq 1.89$, $Q\simeq 0.63$), in which case the critical line reaches the point $K_w=0$, $K=\infty$, and the fixed point B occurs at $\tanh K \simeq 0.70$, $\tanh K_w \simeq 0.02$, with $y(B)\simeq 0.22$, $\tanh K_c \simeq 0.31$. For $b-l>2$, R remains infinite for $K_w=0$, and the critical properties are the same as for finite K_w [flows at criticality go to the fixed point B, Fig.

3(b)]. For $b=7$, $l=3$ ($D\simeq 1.90$, $Q\simeq 0.71$), Fig. 3(b) yields $\tanh K_c \simeq 0.19$, $y(B)\simeq 0.52$. As a rule, except for $b=3$, we find that both T_c and $y=1/\nu$ decrease with increasing b at fixed $b-l$, or with increasing l at fixed b (i.e., decreasing D and Q). T_c and y also increase with Q for fixed D.[14]

Finally, we modified the Sierpiński scheme by keeping D and Q (i.e., b and l) fixed and decreasing the lacunarity,[9] as shown in Fig. 2(b). The resulting flow diagram is *very different* from Fig. 3(b), and in fact has the shape of Fig. 3(a) [with $\tanh K_c \simeq 0.24$, $y(B)\simeq 0.46$]. The values of T_c and y decrease, and the line $K_w=0$ again corresponds to finite R, i.e., $T_c=0$.

Two of us (Y. G. and A. A.) gratefully acknowledge stimulating discussions with B. I. Halperin, D. R. Nelson, M. E. Fisher, S. Fishman, D. Blankschtein, and especially A. N. Berker and Y. Shapir, and the hospitality of B. I. Halperin at Harvard University. This work was supported by the National Science Foundation under Grant No. DMR-77-10210 and by the U. S.-Israel Binational Science Foundation.

[(a)] On leave from IBM Thomas J. Watson Research Center, Yorktown Heights, N. Y. 10598.

[(b)] On leave from Tel Aviv University. Also at the Massachusetts Institute of Technology, Cambridge, Mass. 02139.

[1] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972); E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976); D. J. Wallace and R. K. P. Zia, Phys. Rev. Lett. 43, 808 (1979).

[2] B. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (Freeman, San Francisco, Cal., 1977), and a forthcoming sequel.

[3] A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys. JETP 42, 743 (1975)]; L. P. Kadanoff, Ann. Phys. 100, 359 (1976).

[4] See, e.g., M. E. Fisher, Rev. Mod. Phys. 46, 597 (1974); A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6, p. 357.

[5] The self-similar fractals to which we limit ourselves are constructed by recursive replacement of segments, triangles, or squares by more complex sets made of smaller segments, triangles, or squares. When the number of subpieces is N and their linear relative scale is b, the fractal dimensionality D is defined by $b^D=N$. For sets imbedded in d-dimensional Euclidean space, $D<d$. See Ref. 2.

[6] A fractal's topological dimensionality, D_T, is an integer satisfying $D \geqslant D_T$. For all curves, however ramified, $D_T=1$. See Ref. 2.

[7] The order of ramification at a point P is intended to measure the smallest number of significant interactions which one must cut in order to isolate an arbitrarily small bounded part of points surrounding P. The best available implementation is the Urysohn-Menger order of ramification, $R(P)$ [K. Menger, *Kurventheorie* (Chelsea, New York, 1932), p. 97]. The maximum and minimum values of R obey $R_{\max} \geqslant 2R_{\min}-2$. When equality prevails, a curve is "quasihomogeneous." When $R_{\max}=R_{\min}$ (requiring $R=2$ or $R=\infty$), a curve is "homogeneous".

[8] When $R=\infty$ we define the connectivity Q as the minimum of the fractal dimensionalities of the "cut" required to isolate a bounded infinite part of the system.

[9] Lacunarity is a notion one of us introduced through the study of galactic distributions [B. B. Mandelbrot, C. R. Acad. Sci. Ser. A 288, 81 (1979)], in order to improve the fit of fractal models. For the present purposes, it suffices to observe that Fig. 2(a) (with a large central hole) is more lacunar than Fig. 2(b) (with many small holes). As these examples suggest, one of the roles of lacunarity is to measure the extent of the failure of a fractal to be translationally invariant.

[10] Earlier work on truncated simplexes [D. R. Nelson and M. E. Fisher, Ann. Phys. 91, 226 (1975); D. Dhar, J. Math. Phys. 18, 577 (1977), and 19, 5 (1978)] addressed related problems, for a limited class of systems with $T_c=0$. The definitions of effective dimensionalities in these references disagree with each other. Ours and that of Nelson and Fisher are in agreement. The truncated tetrahedron studied by Nelson and Fisher also demonstrates a system with $R>2$ and $1/\nu=y=0$.

[11] H. E. Stanley, R. J. Birgenau, P. J. Reynolds, and J. F. Nicoll, J. Phys. C 9, L553 (1976); B. B. Mandelbrot, Ann. Israel Phys. Soc. 2, 226 (1978); D. Stauffer, Phys. Rep. 54, 1 (1979); S. Kirkpatrick, *Les Houches Summer School on Ill Condensed Matter 1978*, edited by R Balian *et al*. (North-Holland, Amsterdam, 1979), p. 323.

[12] Fig. 1(d) with equal-length bonds cannot be imbedded in the plane, only in $d=3$.

[13] Note that this value of D is close to that of the backbone in two-dimensional percolation (See Kirkpatrick, Ref. 11).

[14] Details will be published elsewhere.

[15] See e.g., the papers by R. B. Griffiths and C. J. Thompson, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 1.

[16] See also P. Suranyi, Phys. Rev. Lett. 38, 1436 (1977), who used a similar approach to obtain approximate results near T_c.

[17] Note that both T_c and y for $K_w=\infty$ (point A) are always higher than for $K_w=K$ (flowing to point B), because of the higher connectivity.