ity a nonlocal potential which could fit neutron elastic scattering data in the energy range from 0.4 to 24 MeV and where the optical-model parameters were energy independent. This is, of course, a remarkable result since we know that the absorption, for instance, is changing quite dramatically by going from 0.4 to 24 MeV incident projectile energy. We believe that these findings of Perey and Buck strongly support our result that there exists a nonlocal potential, which is not explicitly energy-dependent, which describes elastic nucleon scattering in a wide energy range. The theory presented in this paper may serve as a convenient tool in deriving such a potential. Actual calculation of an energy-independent optical-model potential as outlined in this paper is in progress.

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Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics

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A new Hamiltonian density formulation of a perfect fluid with or without a magnetic field is presented. Contrary to previous work the dynamical variables are the physical variables, ρ, $\vec{v}$, $\vec{B}$, and s, which form a noncanonical set. A Poisson bracket which satisfies the Jacobi identity is defined. This formulation is transformed to a Hamiltonian system where the dynamical variables are the spatial Fourier coefficients of the fluid variables.

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Several advantages may be gained from expressing a set of equations in Hamiltonian form. In addition to their formal elegance, Hamiltonian systems possess Poincaré invariants that influence the dispersion of an ensemble of systems with clustered initial conditions. A manifestly Hamiltonian formulation of a given problem makes it easier to find those approximations that preserve the Hamiltonian character. Here we present such a formulation of hydrodynamics and magnetohydrodynamics.

Hamiltonian systems are most elegant when expressed in canonical coordinates. Hydrodynamics is most usefully expressed in Eulerian variables. These two desiderata conflict. In practice, the penalty paid for adopting noncanonical coordinates is not severe, so that branch of the dichotomy is pursued here.

Previously, the equations of hydrodynamics[1] and magnetohydrodynamics,[2] in both Eulerian and Lagrangian form, have been shown to arise from a suitable Hamilton's principle. Such a Lagrangian density formulation is the natural starting place for derivation of a Hamiltonian density description.[3] Typically, the Euler-Lagrange equation is the fluid equation of motion; the remaining fluid equations have the role of constraints. A Hamiltonian density formulation obtained by Legendre transformation necessarily embodies this division of roles. Alternatively, Hamiltonian-type equations have been given directly for a fluid[4] and for ideal magnetohydrodynamics.[5] In these formulations, Clebsch or other nonphysical variables are necessary and entropy convection is not included. Our formulation departs from previous work in that all of the fluid equations are, in principle, placed on equal footing; further, the dynamical variables are the physical variables. The fluid equations, including entropy convection and (but not necessarily) the Maxwell induction equation, are obtained in Poisson-bracket form; the Hamiltonian density is the energy density of the fluid. The physical variables are noncanonical; this results in alteration of the usual Poisson bracket. The use of noncanonical variables has proven to be fruitful for Hamiltonian systems[6] and a Poisson bracket similar to ours has been used to express the Korteweg-de Vries equation as a Hamiltonian system.[7,8]

In the following, we obtain three essentially equivalent forms for the Poisson bracket. The first, Eq. (6), is expressed in terms of the usual physical variables. The second, Eq. (9), use conserved densities as variables. This form possesses greater symmetry, and facilitates Fourier transformation. In the last form, Eq. (14), the dynamical variables are spatial Fourier coefficients.

We wish to cast the following set of equations into Hamiltonian form:

$$\vec{v}_t = -\nabla(v^2/2) + \vec{v}\times(\nabla\times\vec{v}) - \rho^{-1}\nabla(\rho^2 U_\rho) + \rho^{-1}(\nabla\times\vec{B})\times\vec{B}, \quad (1)$$

$$\rho_t = -\nabla\cdot(\rho\vec{v}), \quad (2)$$

$$\vec{B}_t = \nabla\times(\vec{v}\times\vec{B}), \quad (3)$$

$$s_t = -\vec{v}\cdot\nabla s. \quad (4)$$

Equation (1) is the hydrodynamic force balance equation for a fluid with density ρ and velocity $\vec{v}$, with the addition of the magnetic body force term $\vec{J}\times\vec{B}$. We have eliminated $\vec{J}$ by making use of Ampere's law: $\vec{J} = \nabla\times\vec{B}$. The internal energy per unit mass, $U(\rho, s)$, is a prescribed function of ρ and the entropy per unit mass,[9] s. The intensive variables, pressure p and temperature T, are obtained from this function $p = \rho^2 U_\rho$ and $T = U_s$. Equation (2) is mass conservation. Equation (3) is the Maxwell induction equation with the electric field eliminated by Ohm's law: $\vec{E} + \vec{v}\times\vec{B} = 0$. Here infinite conductivity is assumed. Equation (4) expresses entropy convection; heat flow is assumed to vanish. The equation $\nabla\cdot\vec{B} = 0$ enters our formulation only as an initial condition.

The energy density of a fluid described by Eqs. (1)–(4) is $H = \frac{1}{2}\rho v^2 + \rho U(\rho, s) + \frac{1}{2}B^2$, where $\frac{1}{2}\rho v^2$ is the kinetic-energy density and the remaining two terms are the internal- and magnetic-energy densities. We take this as our Hamiltonian density and construct the Hamiltonian $\hat{H}[\rho, s, \vec{v}, \vec{B}] = \int_V H(\rho, s, \vec{v}, \vec{B})\, d\tau$, where the square brackets are used to indicate that $\hat{H}$ is a functional of the enclosed functions. The integration is over a fixed spatial region V. We desire a Poisson bracket, such that Eqs. (1)–(4) can be represented in the form

$$\bar{\chi}_t{}^i = [\bar{\chi}^i, \hat{H}], \quad i = 0, 1, 2, \ldots 7, \quad (5)$$

where the $\bar{\chi}^i$ are suitable functional dynamical variables.

Before writing this bracket [Eq. (6) below], we briefly discuss the structure of our formulation. Quite generally consider the vector space V, over the real numbers R, whose elements are functionals of the form

$$\hat{F}[\vec{\chi}] = \int_V F(\vec{x}, t; \vec{\chi}, \partial\vec{\chi}/\partial x_\alpha, \partial^2\vec{\chi}/\partial x_\alpha \partial x_\beta, \ldots)\, d\tau,$$

where $\vec{\chi}$ is an n-uple of $C^\infty(V)$ functions $\chi^i(\vec{x}, t)$. [In particular, $\chi^0 \equiv \rho$, $\chi^1 \equiv s$, $(\chi^2, \chi^3, \chi^4) \equiv \vec{v}$, and $(\chi^5, \chi^6, \chi^7) \equiv \vec{B}$. The notation $\partial\vec{\chi}/\partial x_\alpha$ is used to indicate that F depends on the derivatives of χ^i with respect to each of the three spatial variables x_α, $\alpha = 1$–3. We assume F has a finite number of arguments and is a C^∞ function in each of them. The bracket we obtain is a bilinear function which maps $V\times V$ to V. In addition, the bracket possesses the following two important properties: (i) $[\hat{F}, \hat{F}] = 0$ for every $\hat{F} \in V$. For V over R, this is equivalent to $[\hat{F}, \hat{G}] = -[\hat{G}, \hat{F}]$ for $\hat{F}, \hat{G} \in V$; (ii) the Jacobi identity[10] $[\hat{E}, [\hat{F}, \hat{G}]] + [\hat{F}, [\hat{G}, \hat{E}]] + [\hat{G}, [\hat{E}, \hat{F}]] = 0$ for every $\hat{E}, \hat{F}, \hat{G} \in V$. A vector space together with a bracket which has the above properties defines a Lie algebra.[11]

Now we introduce the following bracket[12]:

$$[\hat{F},\hat{G}]=-\int_V\left(\left[\frac{\delta\hat{F}}{\delta\rho}\nabla\cdot\frac{\delta\hat{G}}{\delta\vec{v}}+\frac{\delta\hat{F}}{\delta\vec{v}}\cdot\nabla\frac{\delta\hat{G}}{\delta\rho}\right]+\left[\frac{\delta\hat{F}}{\delta\vec{v}}\cdot\left(\frac{(\nabla\times\vec{v})}{\rho}\times\frac{\delta\hat{G}}{\delta\vec{v}}\right)\right]\right.$$
$$+\left[\rho^{-1}\nabla s\circ\left(\frac{\delta\hat{F}}{\delta s}\frac{\delta\hat{G}}{\delta\vec{v}}-\frac{\delta\hat{G}}{\delta s}\frac{\delta\hat{F}}{\delta\vec{v}}\right)\right]$$
$$\left.+\left\{\rho^{-1}\frac{\delta\hat{F}}{\delta\vec{v}}\cdot\left[\vec{B}\times\left(\nabla\times\frac{\delta\hat{G}}{\delta\vec{B}}\right)\right]+\frac{\delta\hat{F}}{\delta\vec{B}}\circ\left[\nabla\times\left(\vec{B}\times\rho^{-1}\frac{\delta\hat{G}}{\delta\vec{v}}\right)\right]\right\}\right)d\tau\equiv\int_V\frac{\delta\hat{F}}{\delta\chi^i}O^{ij}\frac{\delta\hat{G}}{\delta\chi^j}d\tau. \qquad (6)$$

Here the notation $\delta\hat{F}/\delta\chi^i$ means the functional derivative with respect to χ^i. Suppose each χ^i contains an additional parameter dependence $\chi^i(\vec{x},\alpha_i,t)$. We define the functional derivative by[8]

$$\frac{\partial\hat{F}}{\partial\alpha_i}=\int_V\frac{\delta\hat{F}}{\delta\chi_i}\frac{\partial\chi^i}{\partial\alpha_i}d\tau \quad \text{(not summed)}. \qquad (7)$$

This functional derivative has the role, in finite-dimensional Hamiltonian systems, of the derivatives with respect to phase coordinates $(\partial F/\partial q_i,\partial F/\partial p_i)$. In systems with finite degrees of freedom the Poisson bracket is written

$$[F,G]=\frac{\partial F}{\partial z^i}J^{ij}\frac{\partial G}{\partial z^j},$$

where the z^i are the phase-space coordinates, $z^i\in\{q_1,\ldots,q_N,p_1,\ldots,p_N\}$. In canonical coordinates the cosymplectic form, J^{ij}, is

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},$$

where I is the unit $N\times N$ matrix. In a canonical system this matrix may be full and depend on the dynamical variables. Clearly, this is the case for our bracket, Eq. (6). The cosymplectic form here is the operator O^{ij} which, in addition to being dependent on the dynamical variables, contains derivatives.

Now we complete the description of our formulation and demonstrate the relationship between this bracket and Eqs. (1)–(4). We define a set $D\subset V$ whose elements are of the form

$$\bar{\chi}^i[\chi^i]=\int_V f_i(\vec{x})\chi^i(\vec{x},t)d\tau,$$

$$i=0,1,2,\ldots,7 \quad \text{(not summed)},$$

where $\chi^i\in C^\infty(V)$ and the f_i are arbitrary functions[13] of $\vec{x}$ alone, which vanish on ∂V. D is thus the set of dynamical variables. Substituting $\bar{\chi}^0$ and $\hat{H}$ into Eq. (5) yields

$$\frac{\partial\bar{\chi}^0}{\partial t}-[\bar{\chi}^0,\hat{H}]$$
$$=\int_V f_0(\vec{x})\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})\right)d\tau=0. \qquad (8)$$

Since $f_0(\vec{x})$ is an arbitrary function, by the Du Bois–Reymond[14] lemma, Eq. (8) implies Eq. (2). Equations (1), (3), and (4) follow, from the remaining dynamical variables of the set D, in a similar manner.

Several features of the bracket defined in Eq. (6) deserve comment. First, the density, ρ, appears in the denominator of several terms. This makes it awkward to evaluate the bracket exactly when polynomial or Fourier representations are used for the dynamical variables. This is easily rectified through a nonlinear transformation described below, and the resulting bracket, in terms of the new variables, has a pleasing form. Second, gradients appear throughout the bracket. This is reminiscent of the bracket used in Hamiltonian theories of the Korteweg–de Vries equation[7,8]

$$[F,G]=\int dx\frac{\delta F}{\delta u}\left(\frac{\partial}{\partial x}\frac{\partial G}{\partial u}\right).$$

Two methods have been used to reduce the Korteweg–de Vries bracket to canonical form. Gardner[8] used a Fourier transform to convert the derivatives to numbers, and then scaled the coefficients to achieve canonical form. Similarly motivated, we also consider Fourier transforms below. In another approach to the Korteweg–de Vries equation, Zakharov and Faddeev[7] used a spectral transform to achieve canonical form. This method may be applicable here.

Our new set of Eulerian variables, which yields an improved Poisson barcket is $\{\rho,\sigma,\vec{M},\vec{B}\}$, where $\sigma=\rho s$ and $\vec{M}=\rho\vec{v}$; σ is the specific entropy and $\vec{M}$ is the momentum density. Substitution of these variables into Eqs. (1)–(4) results in eight conservation equations. The pressure is now determined by $p=\rho^2(\tilde{U}_\rho+\sigma\rho^{-1}\tilde{U}_\sigma)$, where $\tilde{U}(\rho,\sigma)=U(\rho,s)$. As a result of the transformation

$$\left.\frac{\delta}{\delta\rho}\right|_{\vec{v},s}=\left.\frac{\delta}{\delta\rho}\right|_{\vec{M},\sigma}+\rho^{-1}\vec{M}\cdot\frac{\delta}{\delta\vec{M}}+\sigma\rho^{-1}\frac{\delta}{\delta\sigma},$$

together with similar transformations for the re-

$$\underline{O}_{\vec{k}\vec{l}} = 2\pi i \left[\begin{array}{cc|c|ccc} 0 & 0 & -\rho_{\vec{l}+\vec{k}}\vec{k} & 0 & 0 & 0 \\ 0 & 0 & -\sigma_{\vec{l}+\vec{k}}\vec{k} & 0 & 0 & 0 \\ \hline \rho_{\vec{l}+\vec{k}}\vec{l} & \sigma_{\vec{l}+\vec{k}}\vec{l} & \vec{l}\,\vec{M}_{\vec{l}+\vec{k}} - \vec{M}_{\vec{l}+\vec{k}}\vec{k} & \multicolumn{3}{c}{\vec{l}\,\vec{B}_{\vec{l}+\vec{k}} - \vec{l}\cdot\vec{B}_{\vec{l}+\vec{k}}\underline{I}'} \\ \hline 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -\vec{k}\,\vec{B}_{\vec{l}+\vec{k}} + \vec{k}\cdot\vec{B}_{\vec{l}+\vec{k}}\underline{I}' & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \end{array}\right]$$

FIG. 1. The cosympletic form $\underline{O}_{\vec{k},\vec{l}}$.

maining variables, Eq. (6) becomes

$$[\hat{F},\hat{G}] = -\int_V d\tau\left(\rho\left(\frac{\delta\hat{F}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{G}}{\delta\rho} - \frac{\delta\hat{G}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{F}}{\delta\rho}\right) + \vec{\mathbf{M}}\cdot\left(\frac{\delta\hat{F}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{G}}{\delta\vec{\mathbf{M}}} - \frac{\delta\hat{G}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{F}}{\delta\vec{\mathbf{M}}}\right)\right.$$
$$\left. +\sigma\left(\frac{\delta\hat{F}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{G}}{\delta\sigma} - \frac{\delta\hat{G}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{F}}{\delta\sigma}\right) + \vec{\mathbf{B}}\cdot\left\{\left(\frac{\delta\hat{F}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{G}}{\delta\vec{\mathbf{B}}} - \frac{\delta\hat{G}}{\delta\vec{\mathbf{M}}}\cdot\nabla\frac{\delta\hat{F}}{\delta\vec{\mathbf{B}}}\right) + \left[\left(\nabla\frac{\delta\hat{F}}{\delta\vec{\mathbf{B}}}\right)\cdot\frac{\delta\hat{G}}{\delta\vec{\mathbf{M}}} - \left(\nabla\frac{\delta\hat{G}}{\delta\vec{\mathbf{B}}}\right)\cdot\frac{\delta\hat{F}}{\delta\vec{\mathbf{M}}}\right]\right\}\right). \quad (9)$$

Notice that each term contains one Eulerian variable in the numerator; the terms in the denominator have been eliminated.

Now consider a transformation of the Hamiltonian coordinates from Eulerian variables to the coefficients of the Fourier transform of these variables. For convenience, we take V to be a unit cube and adopt periodic boundary conditions. Then

$$\rho = \sum_{\vec{k}}\rho_{\vec{k}}(t)\exp(2\pi\vec{k}\cdot\vec{x}), \quad (10)$$

where $k \in Z\times Z\times Z$ (Z, the integers). We observe from Eq. (7) that

$$\frac{\partial\hat{F}}{\partial\rho_{\vec{k}}} = \int_V \frac{\delta\hat{F}}{\delta\rho}\exp(2\pi i\vec{k}\cdot\vec{x})\,d\tau. \quad (11)$$

Inverting Eq. (11), we obtain

$$\frac{\delta\hat{F}}{\delta\rho} = \sum_{\vec{k}}\frac{\partial\hat{F}}{\partial\rho_{\vec{k}}}\exp(-2\pi i\vec{k}\cdot\vec{x}). \quad (12)$$

Inserting Eqs. (10) and (12), and the analogous expressions for the other variables in our set, into Eq. (9) yields

$$[\hat{F},\hat{G}] = \sum_{\vec{k},\vec{l}}\frac{\partial\hat{F}}{\partial\vec{z}_{\vec{k}}}\cdot\underline{O}_{\vec{k},\vec{l}}\cdot\frac{\partial\hat{G}}{\partial\vec{z}_{\vec{l}}}, \quad (13)$$

where $\vec{z}_{\vec{k}}$ is the octuple $(\rho_{\vec{k}},\sigma_{\vec{k}},\vec{M}_{\vec{k}},\vec{B}_{\vec{k}})$, and the matrix $\underline{O}_{\vec{k},\vec{l}}$ is shown in Fig. 1. Here $\underline{I}$ is the 3×3 unit matrix appropriate to the box in which it is contained. The matrix has the important property $\underline{O}_{\vec{k},\vec{l}} = -\underline{\tilde{O}}_{\vec{l},\vec{k}}$, where the tilde indicates transpose. Equation (13) can be written as follows:

$$[F,G] = \frac{\partial F}{\partial z^i}J^{ij}\frac{\partial G}{\partial z^j}, \quad i,j\in z, \quad (14)$$

where $z^i \in \{\rho_{\vec{k}},\sigma_{\vec{k}},\vec{M}_{\vec{k}},\vec{B}_{\vec{k}} \mid \vec{k}\in Z\times Z\times Z\}$. The matrix $\underline{J}$ has the property $J^{ij} = -J^{ji}$ and its elements can be obtained by a suitable map[15] of the indices of $\underline{O}_{\vec{k},\vec{l}}$ onto Z. Clearly Eq. (14) is of the same form as finite Hamiltonian systems, but here J is of infinite order. Approximation techniques, along with the proof of the Jacobi identity, integral invariants, and commutation relations, will be the subject of a future publication.

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[10]We have proved the Jacobi identity for functionals $\hat{F}, \hat{G}, \hat{E}$ with corresponding F, G, E which are functions of $\vec{x}$, t, and $\vec{\chi}$. This is sufficient for our purposes here. The more general case is of interest; e.g., the dynamical variable with the integrand $\nabla \times \vec{v}$ yields, when substituted into Eq. (5), the vorticity equation.

[11]V. Arnold, *Mathematical Methods of Classical Mechanics* ((Springer-Verlag, New York, 1978), p. 208; R. Bishop and R. Crittendon, *Geometry of Manifolds* (Academic, New York, 1964), p. 26.

[12]The hydrodynamic bracket is obtained by setting $B = 0$, isentropic hydrodynamics by further setting $s = 0$, and irrotational isentropic hydrodynamics by finally setting $\nabla \times \vec{v} = 0$. The Jacobi identity, as proved, is satisfied for these cases. (We use repeated index notation in the second equality here and henceforth, except where noted.) Also note that in order to see that this expression is antisymmetric, integrations by parts must be performed. The surface terms obtained are assumed to vanish; this is an added restriction on the elements of the vector space V. For V composed of $\hat{H}$ and $\{\overline{\chi}^i\}$, the terms vanish by virtue of the arbitrary functions $f_i(\vec{x})$. This imposes no restrictions on boundary conditions.

[13]Conventionally, the functional derivative of a function with respect to itself is seen to be the Dirac δ function. For $f_i(\vec{x}) \equiv \delta(\vec{x} - \vec{x}')$, this is obtained.

[14]D. Smith, *Variational Methods in Optimization* (Prentice-Hall, Englewood Cliffs, N. J., 1974), p. 357.

[15]A bijection f: $S \times Z \times Z \times Z \rightarrow Z$, where $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is needed for this purpose. A useful example will be given in another paper.