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Quantum-Mechanical Radiation-Pressure Fluctuations in an Interferometer

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The interferometers now being developed to detect gravitational vaves work by measuring small changes in the positions of free masses. There has been a controversy whether quantum-mechanical radiation-pressure fluctuations disturb this measurement. This Letter resolves the controversy: They do.

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An interferometer is a particularly promising type of gravitational-wave detector. In the early 1970's a group at Hughes Research Laboratories in Malibu, California, searched unsuccessfully for gravitational waves, using an interferometer of modest sensitivity.[1] Now interferometers of dramatically improved sensitivity are being developed in several laboratories around the world.[2-4]

The prototypical interferometer for gravitational-wave detection is a multireflection Michelson system of the sort sketched in Fig. 1.[5] An idealized version of such an interferometer works as follows. Light enters the interferometer from a laser, is split at a lossless, 50-50 beam splitter, bounces back and forth many times between perfectly reflecting mirrors in the nearly equal-length arms of the interferometer, and finally is recombined at the beam splitter. The end mirrors are attached to large masses, each of mass m. The beam splitter and the inner mirrors are rigidly attached to one another and to a mass M. For simplicity I assume $M \gg m$, so that the beam splitter can be regarded as stationary in an inertial frame.

This standard type of interferometer measures the difference $z \equiv z_2 - z_1$ between the positions of the end mirrors. The intensity in either of the output ports, measured by some photodetector (e.g., a photodiode), provides a direct measure of the phase difference $\delta\Phi$ between the light in the two arms of the interferometer. This, in turn, is related to z by $\delta\Phi = 2b\omega z/c$, where ω is the light's angular frequency and b is the number of reflections at each end mirror. A classical

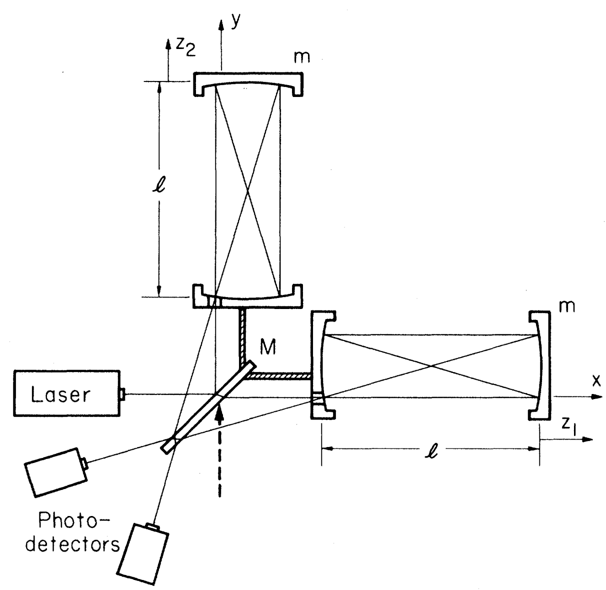

FIG. 1. Schematic of Michelson interferometer ($b = 2$) described in text.

force acting on the end masses (e.g., a gravitational wave) is detected by the changes it produces in z.

Quantum mechanics places a limit on the accuracy of any measurement of the position of a free mass. The standard interferometer cannot escape this limit. In a measurement of duration τ, the minimum possible error in its determination of z is the "standard quantum limit":

$$(\Delta z)_{\rm SQL} = (2\hbar\tau/m)^{1/2}, \tag{1}$$

a limit which follows immediately from the Heisenberg uncertainty principle.[3,6,7] The standard quantum limit determines the minimum possible gravitational-wave amplitude h the interferometer can detect: $h_{\rm SQL} \sim (\Delta z)_{\rm SQL}/l$, where l is the interferometer's arm length. For ground-based interferometers, restricted by seismic noise to $\tau \lesssim 0.1$ sec, the quantum limit may not be a serious problem, provided l can be made large enough ($h_{\rm SQL} \sim 10^{-22}$ for $\tau \simeq 10^{-2}$ sec, $m \simeq 10^5$ g, $l \simeq 1$ km). However, for a space-based interferometer of modest size ($m \sim 10^6$ g, $l \sim 1$ km), aimed at lower frequencies ($\tau \sim 10^4$ sec), the quantum limit might be a serious problem.

The quantum limit (1) has also been obtained from an argument which balances the error due to photon-counting statistics against the disturbance of the end mirrors' positions produced by fluctuating radiation pressure.[3,5,8,9] This argument runs as follows: The laser has a mean power P, so that during a measurement time τ the mean number of photons which pass through the interferometer is $N = P\tau/\hbar\omega$. The laser power fluctuates, and the fluctuations produce an uncertainty in N at least as large as $\Delta N \sim N^{1/2}$. The resulting uncertainty in the number of output photons restricts one's determination of $\delta\Phi$ to an accuracy $\Delta(\delta\Phi) \sim N^{-1/2}$, which corresponds to a photon-counting error in z given by

$$(\Delta z)_{pc} = (c/2b\omega)\Delta(\delta\Phi) \sim (c/2b\omega)N^{-1/2}. \tag{2}$$

The mean radiation-pressure forces on the end masses do not affect one's measurement, because they do not change z. However, the fluctuations in laser power also produce radiation-pressure forces on the end masses. If the fluctuations in the two arms are *uncorrelated*, they will disturb the *difference* $p \equiv p_2 - p_1$ in the end masses' momenta. During the measurement time this disturbance produces an uncertainty $\Delta p \sim (2\hbar\omega b/c)N^{1/2}$, which leads to a radiation-pressure uncertainty in z given by

$$(\Delta z)_{rp} \sim (\Delta p)\tau/2m \sim (\hbar\omega b/c)(\tau/m)N^{1/2}. \tag{3}$$

The total error in z is $\Delta z = [(\Delta z)_{pc}^2 + (\Delta z)_{rp}^2]^{1/2}$. Minimizing Δz with respect to N (or P) yields (i) a minimum error of order of the standard quantum limit and (ii) an optimum power at which the minimum error is achieved[8]:

$$P_{\rm opt} \sim \tfrac{1}{2}(mc^2/\tau)(1/\omega\tau)(1/b^2) \simeq 300 \text{ W} \tag{4}$$

for $m \simeq 10^5$ g, $\tau \simeq 10^{-2}$ sec, $\omega \simeq 4\times 10^{15}$ rad-sec^{-1}, $b \simeq 200$.

The existence of the standard quantum limit (1) is firm, but the above argument leading to it and to $P_{\rm opt}$ has been under suspicion since before it was first advanced.[5] It relies critically on the assumption that the laser-power fluctuations are uncorrelated in the two arms. Correlated fluctuations drive only the sum of the end masses' momenta; in principle they do not affect one's measurement of z. It has always been a mystery why a perfect beam splitter would split the incident fluctuations unequally. The result has been a "lively but unpublished controversy"[4] over the existence of a fluctuating radiation-pressure force which drives p, and consequently over the existence of an optimum laser power.

Here I resolve this controversy. The resolution is based on a rigorous quantum-mechanical analysis of an interferometer, the salient features of which are sketched below. The key results of this analysis are the following: *There is a fluctuating radiation-pressure force which drives p. However, it has nothing to do with fluctuations in laser power; rather, it is an intrinsic property of a standard interferometer.*

The rigorous analysis reveals two different, but equivalent, points of view on the origin of the relevant radiation-pressure fluctuations.

The first point of view identifies the beam splitter as the culprit. Suppose N photons are incident on the beam splitter. It scatters each photon independently, thereby producing a binomial distribution of photons in each arm of the interferometer. The two binomial distributions are precisely *anticorrelated* (if too many photons go down one arm, too few go down the other), and this is precisely what is necessary to produce a "$\sqrt{N}$" fluctuating force which drives p. This point of view has been suggested previously by Edelstein *et al.*[9]

The second point of view focuses on vacuum (zero-point) fluctuations in the electromagnetic field. Suppose some light enters the interferometer from the port opposite the laser port (direction of dashed arrow in Fig. 1). If this light has the right phase to increase the intensity in one arm, it decreases the intensity in the other arm. Thus such light automatically produces a force which drives p. Although there is no light source which injects radiation from this port, there are inevitably vacuum fluctuations incident from this direction. These vacuum fluctuations, when superposed on the light from the laser, produce the required "$\sqrt{N}$" fluctuating force.

Either of these points of view, when extended to follow the light all the way through the interferometer, shows that the photon-counting error in determining $\delta\Phi$ $[\Delta(\delta\Phi)\sim N^{-1/2}]$ is also an intrinsic property of a standard interferometer —i.e., the interferometer itself produces the required uncertainties in the number of output photons. In principle the photon-counting error has nothing to do with laser-power fluctuations, whose effect is negligible very close to a null in the fringe pattern. In most analyses the photon-counting error is attributed to shot noise in the photodetectors[1,4,5,8,9]; these analyses implicitly assume the required uncertainties in the number of photons striking the photodetectors—uncertainties here revealed to be intrinsic to the interferometer.

Turn now to a simplified analysis which demonstrates how the two points of view arise. In this simplified analysis one ignores the entire interferometer, except the beam splitter and its initial scattering of the light; one assumes that the beam splitter is infinitely long; and one considers only four (plane-wave) modes of the electromagnetic field in the presence of the beam splitter. A complete analysis requires constructing a finite wave packet and following it entirely through the interferometer. I have carried out such a complete analysis; it does not alter the results obtained here. Unruh[10] has recently sketched a procedure for carrying out a complete analysis.

The first two modes of interest are "in" states in the sense of scattering theory. "In" states are modes which are appropriate for constructing precollision wave packets that scatter off the beam splitter. The first mode (mode 1^+) consists of (i) an incident plane wave with angular frequency ω, propagating inward along the x axis (light incident from the laser port; see Fig. 1), and (ii) scattered waves propagating along the two arms of the interferometer. Outside the beam splitter the electric field of mode 1^+ has the form

$$\vec{E}_1{}^+ = \vec{e}_z \times \begin{cases} Ae^{i(kx-\omega t)} + i2^{-1/2}e^{i\Delta}Ae^{i(ky-\omega t)}, & y > x, \\ 2^{-1/2}e^{i\Delta}Ae^{i(kx-\omega t)}, & y < x. \end{cases} \tag{5}$$

Here $k = \omega/c$, A is a (real) constant determined by one's choice of normalization, and Δ is a property of the beam splitter. The 90° phase difference between the waves in the two arms is dictated by the symmetries of the beam splitter.

The second mode (mode 2^+) is the reflection of mode 1^+ through the beam splitter—i.e., it is the "in" mode whose incident wave propagates inward along the y axis (light incident from direction of dashed arrow in Fig. 1). Its electric field has the form

$$\vec{E}_2{}^+ = \vec{e}_z \times \begin{cases} 2^{-1/2}e^{i\Delta}Ae^{i(ky-\omega t)}, & y > x, \\ Ae^{i(ky-\omega t)} + i2^{-1/2}e^{i\Delta}Ae^{i(kx-\omega t)}, & y < x. \end{cases} \tag{6}$$

The third and fourth modes (modes 1^- and 2^-) are "out" states (time-reversed "in" states). "Out" modes are appropriate for constructing postcollision wave packets. Modes 1^- and 2^- are the "out" states whose exiting plane waves propagate along the x and y axes, respectively. Their electric fields are related to the "in" modes by

$$\vec{E}_1{}^- = 2^{-1/2}e^{-i\Delta}(\vec{E}_1{}^+ - i\vec{E}_2{}^+), \tag{7a}$$

$$\vec{E}_2{}^- = i2^{-1/2}e^{-i\Delta}(\vec{E}_1{}^+ + i\vec{E}_2{}^+). \tag{7b}$$

Let the creation and annihilation operators for modes 1^+ and 2^+ be denoted by $\hat{a}_1{}^\dagger$, $\hat{a}_1$ and $\hat{a}_2{}^\dagger$, $\hat{a}_2$;

similarly, for modes 1^- and 2^-, $\hat{b}_1^\dagger$, $\hat{b}_1$ and $\hat{b}_2^\dagger$, $\hat{b}_2$. Equations (7) imply

$$\hat{b}_1 = 2^{-1/2} e^{i\Delta}(\hat{a}_1 + i\hat{a}_2), \tag{8a}$$

$$\hat{b}_2 = i2^{-1/2} e^{i\Delta}(\hat{a}_1 - i\hat{a}_2). \tag{8b}$$

Now consider the case in which precisely N photons are incident on the beam splitter from the laser port. Then the state $|\Psi\rangle$ of the electromagnetic field is an N-photon excitation of mode 1^+: $|\Psi\rangle = (N!)^{-1/2}(\hat{a}_1^\dagger)^N|0\rangle$, where $|0\rangle$ is the photon vacuum. To see how this state looks in the arms of the interferometer, one decomposes it with respect to the "out" modes:

$$|\Psi\rangle = e^{iN\Delta}\sum_{l=0}^{N} i^{N-l}\left[\frac{1}{2^N}\binom{N}{l}\right]^{1/2}\frac{(\hat{b}_1^\dagger)^l(\hat{b}_2^\dagger)^{N-l}}{[l!(N-l)!]^{1/2}}|0\rangle. \tag{9}$$

This decomposition explicitly displays the anticorrelated binomial distributions of the first point of view.

The second point of view comes from considering the operator $\mathcal{P} \equiv (2\hbar\omega b/c)(\hat{b}_2^\dagger\hat{b}_2 - \hat{b}_1^\dagger\hat{b}_1)$, which specifies the difference in the momenta transferred to the end masses. When rewritten in terms of operators for the "in" modes, $\hat{\mathcal{P}} = i(2\hbar\omega \times b/c)(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2)$ is clearly due to interference of modes 1^+ and 2^+. In the state $|\Psi\rangle$, $\langle\hat{\mathcal{P}}\rangle = 0$ but $\langle\hat{\mathcal{P}}^2\rangle^{1/2} = (2\hbar\omega b/c)N^{1/2}$. The nonzero value of $\langle\hat{\mathcal{P}}^2\rangle$ is due to the zero-point excitation of mode 2^+.

A parameter $\beta \equiv \langle\hat{b}_1^\dagger\hat{b}_1\hat{b}_2^\dagger\hat{b}_2\rangle/\langle\hat{b}_1^\dagger\hat{b}_1\rangle^2$ characterizes the degree of correlation of the number of photons in the two arms of the interferometer. Correlation corresponds to $\beta > 1$; anticorrelation to $\beta < 1$, a situation referred to as "photon antibunching."[11] The N-photon state $|\Psi\rangle$ has $\beta = 1 - 1/N$. An uncertainty ΔN in the number of incident photons—due to power fluctuations in the light source—increases β to $\beta = 1 - 1/N + (\Delta N/N)^2$ (e.g., a coherent state of mode 1^+ has $\Delta N = N^{1/2}$ and $\beta = 1$). However, even when $\beta > 1$, the relevant quantity z is perturbed by the anticorrelated, $-1/N$ contribution to β. As the second point of view makes clear, the anticorrelation can be understood by imagining that mode 2^+ is randomly excited, with its rms power ($\sim\hbar\omega/\tau$) within the bandwidth of interest ($\Delta\omega \simeq \pi/\tau$) determined by the size of zero-point fluctuations.

The disturbance of z by radiation pressure is an example of the "back-action" disturbance of a quantity by the apparatus which measures it.[7] For measurements of the position of a free mass, back action is unavoidable; it enforces the standard quantum limit (1). It has been thought that back action could be understood as a consequence of noise fed back onto the measured quantity from an amplifier.[7] If the amplifier were noiser than the minimum permitted by quantum mechanics, then the back action would also exceed the minimum value. For the interferometer of Fig. 1—indeed, for any interferometer whose input and output ports do not coincide—the situation is different. Light from the photodetectors disturbs z far less than the radiation-pressure uncertainty (3), provided $P_{d1}P_{d2} \ll P\hbar\omega/\tau$, where P_{d1} and P_{d2} are the rms powers emitted by the photodetectors within a bandwidth $\simeq \pi/\tau$ about ω; even if the photodetectors and subsequent electronics were exceedingly noisy, the back action would remain essentially at the minimum level set by vacuum fluctuations.

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Random-Energy Model: Limit of a Family of Disordered Models

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In this Letter, a simple model of disordered systems—the random-energy model—is introduced and solved. This model is the limit of a family of disordered models, when the correlations between the energy levels become negligible. The properties are qualitatively the same as those of the Sherrington-Kirkpatrick model. Moreover, this random-energy model looks like a simple approximation to any spin-glass model.

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Recently, a lot of effort has been devoted to solving the Sherrington-Kirkpatrick model (the S.K. model).[1] This model was introduced to enable one to understand the properties of the Edwards-Anderson spin-glass model[2] in the case where the range of the interactions is infinite and where, therefore, a mean-field theory[3] for spin glass models would be exact. Though it is now accepted that the failure of the solution initially proposed is due to a breaking of symmetry in the replica space,[4] and in spite of the effort displayed[5,6] to find new solutions of the S.K. model, no simple analytic solution has yet been proposed.

In this Letter, I introduce and solve a new model of disordered systems, the random-energy model. This model describes a system whose energy levels are independent random variables. Many of its properties are very similar to those of the S.K. model: the same qualitative phase diagram, the same free energy in the high temperature phase, the same kind of corrections to the thermodynamic limit. It gives a simplified picture of a transition in a disordered system: The specific heat vanishes in the whole low-temperature phase and the system becomes completely frozen below its critical temperature.

This random-energy model is the limit of a family of models with random interactions which generalize the S.K. model. To describe these models, we consider a system of N interacting Ising spins with infinite-ranged random p-spin interactions. For such a model, the Hamiltonian $\mathcal{H}_p$ can be written

$$\mathcal{H}_p(\{\sigma\}) = -\sum_{(i_1, i_2, \ldots, i_p)} A_{i_1 i_2 \ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \quad (1)$$

where in $\mathcal{H}_p$, there is a random interaction $A_{i_1\ldots i_p}$ for any group of p spins in the system. In order to ensure an extensive thermodynamic limit, one has to scale properly the probability distribution of the interactions $A_{i_1\ldots i_p}$ with N. I choose here Gaussian distributions,

$$\rho(A_{i_1\ldots i_p}) = (N^{p-1}/\pi J^2 p!)^{1/2}\exp[-(A_{i_1\ldots i_p})^2 N^{p-1}/J^2 p!], \quad (2)$$

where $p=1$ corresponds to a system of free spins in a random magnetic field, and $p=2$ is the S.K. model.

To establish the relation between all these models, I introduce the one-level probability distribution $P(E)$ as the probability that a given configuration of the spins $\{\sigma_i^{(1)}\}$, say configuration (1), has a given energy:

$$P(E) = \langle\delta(E - \mathcal{H}(\{\sigma^{(1)}\}))\rangle, \quad (3)$$

where in (3) the average is taken over all the possible choices of the interactions $A_{i_1\ldots i_p}$. In the same way, one can define the two-level probability distribution $P(E_1, E_2)$ as the probability that two given

