Optical Turbulence: Chaotic Behavior of Transmitted Light from a Ring Cavity

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It is theoretically shown that the transmitted light from a ring cavity containing a nonlinear dielectric medium undergoes transition from a stationary state to periodic and nonperiodic states, when the intensity of the incident light is increased. The nonperiodic state is characterized by a chaotic variation of the light intensity and associated broadband noise in the power spectrum. The experimental possibility of observing such a transition is also discussed.

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Bistability seen in the optical transmission of a cavity filled with a nonlinear medium has acquired much attention from its applicability as an optical device[1-3] as well as from the theoretical side since it offers a typical example of the first-order phase transition in systems far from equilibrium.[4,5] Recently the possibility was pointed out that the transmitted light from such a cavity exhibits periodic self-pulsing under suitable conditions.[6]

On the other hand, it is being recognized as a fairly universal fact that nonlinear systems far from equilibrium undergo a sequence of transitions from a stationary state to periodic, and finally to nonperiodic states, when parameter values involved are varied.[7,8] Such transitions will be found in various areas of the natural sciences,[9,10] as exemplified by the transition from laminar flow to turbulent flow in fluid systems.[11] It is therefore reasonable to expect that the bistable optical system is also a candidate for investigating such a transition, which will be demonstrated theoretically below.

Let us consider a ring cavity containing a nonlinear dielectric medium of length l, as illustrated in Fig. 1(a).[5,12] Mirrors 1 and 2 have reflectivity R, while mirrors 3 and 4 have 100% reflectivity, so that a part of the transmitted light is fed back to the medium. The slowly varying (complex) envelope of the electric field at time t and position z in the cavity, $\hat{E}(t,z)$, satisfies the boundary condition

$$\hat{E}(t,0) = (1-R)^{1/2}\hat{E}_I + R\exp(ikL)\hat{E}(t-(L-l)/c, l), \quad (1)$$

where $\hat{E}_I$ is the envelope of the electric field of the incident light, k its wave number, and L the cavity length.

We assume that the response of the medium is described by the Debye relaxation equation. Under the boundary condition (1), the Maxwell-Debye equations which govern the dynamics of the system are integrated with respect to the space variable and lead to the coupled differential-difference equations[12,13]

$$E(t) = A + BE(t-t_R)\exp\{i[\varphi(t) - \varphi_0]\}, \quad (2a)$$

$$\gamma^{-1}\dot{\varphi}(t) = -\varphi(t) + \mathrm{sgn}(n_2)\,|E(t-t_R)|^2. \quad (2b)$$

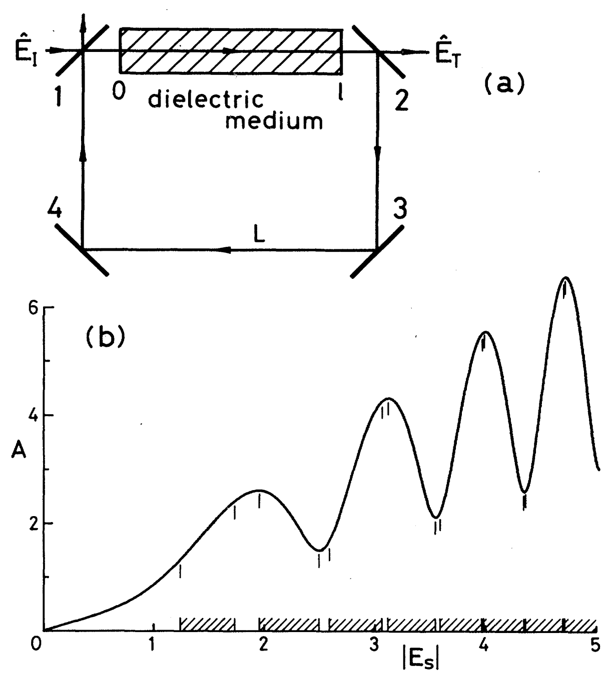

FIG. 1. (a) A ring cavity containing a nonlinear dielectric medium. (b) $|E_S|$ vs A relation for $B = 0.4$ and $\varphi_0 = 0$. The states corresponding to the shaded ranges on the $|E_S|$ axis are unstable.

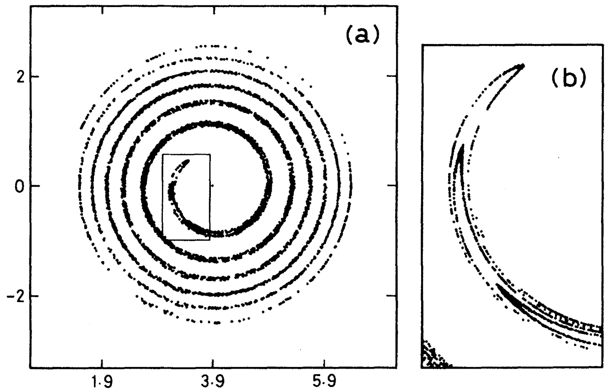

FIG. 2. (a) Plot of 5000 successive points of a series E_n on the complex E plane. The parameter values chosen are $B = 0.4$, $A = 3.9$, and $\varphi_0 = 0$. (b) Enlargement of the rectangular region of (a).

Here the electric field is scaled in a dimensionless form by $E(t) = \{k|n_2|(1-e^{-\alpha l})/\alpha\}^{1/2}\hat{E}(t,0)$, where α is the absorption coefficient of the medium and n_2 the quadratic coefficient of the nonlinear refractive index. The intensity of the transmitted light is given by $(1-R)\exp(-\alpha l)|E(t-l/c)|^2$. Variable $\varphi(t)$ in Eqs. (2a) and (2b) denotes the phase shift suffered by the electric field in the medium, and φ_0 is a mistuning parameter of the cavity. γ is the Debye relaxation rate. Parameter A is defined by $A \equiv \{(1-R)k|n_2|(1-e^{-\alpha l})/\alpha\}^{1/2}|\hat{E}_I|$, which is proportional to the amplitude of the incident field. Parameter B, defined by $B \equiv R\, e^{-\alpha l/2}$ (<1), characterizes the dissipation of the electric field in the cavity. Time delay $t_R \equiv L/c$ originates from the propagation of light. Hereafter we confine ourselves to the case of $n_2 > 0$.[14]

The stationary solutions of Eqs. (2a) and (2b), denoted by E_s, are given as a multivalued function of A, satisfying

$$[1+B^2-2B\cos(|E_s|^2-\varphi_0)]|E_s|^2 = A^2, \tag{3}$$

which is illustrated in Fig. 1(b). In the limit of $1-B \ll 1$, $\varphi_0 \ll 1$, and $A \ll 1$, Eq. (3) is reduced to the bistability relation discussed by previous authors.[1,2]

These E_s are not always stable. To demonstrate this, let us first consider the limit of $t_R\gamma \to \infty$, where the medium responds to the electric field adiabatically. In this limit, Eqs. (2a) and (2b) are reduced to the difference equation

$$E(t) = A + BE(t-t_R)\exp\{i[|E(t-t_R)|^2-\varphi_0]\} \equiv U[E(t-t_R)]. \tag{4}$$

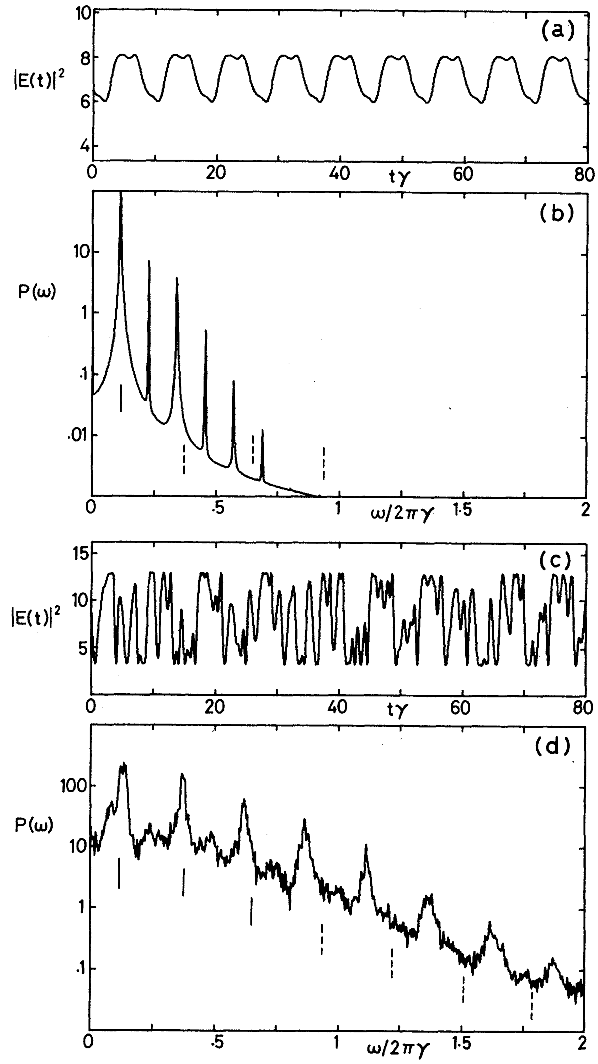

FIG. 3. $|E|^2$ vs time for $B = 0.3$, $\tau_R = 3.5$, $\varphi_0 = 0$, and (a) $A = 2.17$, and (c) $A = 2.85$; power spectrum of $|E|^2$ for $B = 0.3$, $\tau_R = 3.5$, $\varphi_0 = 0$, and (b) $A = 2.17$, and (d) $A = 2.85$. The solid and broken vertical lines in (b) and (d) indicate $|\mathrm{Im}(\lambda)|/2\pi\gamma$ of the unstable and stable modes, respectively.

A linear stability analysis reveals that the stationary solutions of Eq. (4) are stable only if $0 < G^{-1} < 2(1+B^2)$, where G is defined by $G \equiv d|E_s|^2/dA^2$. Considering that $G^{-1} \sim 2A^2B\sin(|E_s|^2-\varphi_0)$ for $A^2 \gg 1$, we know that most of the stationary solutions are unstable for $A^2B \gg 1$ [Fig. 1(b)].

What solution then appears when E_s becomes unstable? By using a computer we have traced the time series $E_n \equiv E(nt_R)$ and have found that, when parameter A is varied in such a way that G^{-1} gets over $2(1+B^2)$, the series E_n undergoes successive bifurcations, forming a periodic set of 2^1 points, 2^2 points, 2^3 points, ..., successively, and finally gets into a chaotic one wandering in an apparently erratic manner. Figure 2(a) shows the plot of 5000 successive points of a series E_n in the chaotic regime, starting from an arbitrarily chosen initial point $E_0 = (3.9, 0)$. The series E_n itself is very sensitive to the choice of the initial point, but the "curves" on which E_n wanders, except for the first few points, are independent of it. In this sense these "curves" may be regarded as a strange attractor.[7,15] The strange attractor in Fig. 2(a) exhibits a complex structure, as seen in Fig. 2(b), but its rough shape is expressed by a spiral

$$|E - A|^2 = B^2[\arg(E - A) + \varphi_0] \qquad (5)$$

for small B, which can be derived analytically. Physically this erratic wandering of E_n may be attributed to large, field-dependent phase shift suffered by the electric field each time it travels around in the cavity.

However, even in the limit $t_R\gamma \to \infty$, the description of $E(t)$ by Eq. (4) is valid only in a limited initial range of time. The reason is as follows. Assume that $E(t)$ slightly fluctuates in the time interval $0 \leq t < t_R$ and let $E(t_1) \neq E(t_2)$ $(0 \leq t_1 \neq t_2 < t_R)$. In the chaotic regime, their n-fold images $U^n[E(t_1)]$ and $U^n[E(t_2)]$ fall on quite different points on the strange attractor. Therefore a slight fluctuation in the initial time interval develops into a wild one, its characteristic time becoming shorter and shorter, with the increase of n. When the characteristic time of the fluctuation has been reduced to the order of γ^{-1}, the adiabatic approximation breaks down and the left-hand side of Eq. (2b) can no longer be neglected. In order to study this stage, we have to return to the original Eqs. (2a) and (2b).

The stability of the stationary solutions of Eqs. (2a) and (2b) can also be determined by a linear stability analysis. The amplification rate λ of a small fluctuation added on the stationary solution satisfies the characteristic equation

$$1 + 2B[-\cos(|E_s|^2 - \varphi_0) + |E_s|^2 \sin(|E_s|^2 - \varphi_0)(\lambda/\gamma + 1)^{-1}]\exp(-t_R\lambda) + B^2\exp(-2t_R\lambda) = 0. \qquad (6)$$

One may regard $i\lambda$ as the eigenmode frequency of the fluctuation. For the sake of simplicity let us confine ourselves to the case of $B \ll 1$ and $A^2B \sim O(1)$. In this case Eqs. (2a) and (2b) are approximately reduced to the one-variable differential-difference equation for $\varphi(t)$,

$$\gamma^{-1}\dot{\varphi}(t) = -\varphi(t) + A^2\{1 + 2B\cos[\varphi(t - t_R) - \varphi_0]\}; \qquad (7)$$

$E(t)$ is given by $|E(t)|^2 = A^2\{1 + 2B\cos[\varphi(t - t_R) - \varphi_0]\}$. Correspondingly, Eq. (6) is also simplified. It is not difficult to investigate this simplified equation and to obtain the following results: For $0 < G^{-1} < 2$, all $\mathrm{Re}(\lambda)$ are negative, so that the stationary solution is stable. For $G^{-1} > 2$, a pair of eigenmodes $i\lambda$ and $i\lambda^*$ becomes unstable each time the dimensionless parameter $\tau_R \equiv t_R\gamma$ exceeds one of the critical values

$$\tau_R^{(n)} = G(1 - 2G)^{-1/2}\{(2n + 1)\pi - \arccos[G(1 - G)^{-1}]\} \quad (n = 0, 1, 2, \ldots). \qquad (8)$$

That is, the stationary solution is stable only if $\tau_R < \tau_R^{(0)}$.

To investigate what occurs when this condition is not satisfied, we have solved Eq. (7) numerically for various numbers of unstable modes, setting B, φ_0, and τ_R at fixed values and varying A. In Fig. 3 the temporal behavior of the intensity of the electric field and its power spectrum are shown for two choices of the parameters. In case only one pair of modes is unstable [Figs. 3(a) and 3(b)], the electric field exhibits a simple periodic behavior and the power spectrum consists of sharp peaks at a fundamental frequency, which is located very close to $|\mathrm{Im}(\lambda)|$ of the unstable modes, and at frequencies of its higher harmonics. As parameter A is varied and the number of unstable modes is increased, a transition from this periodic state to a turbulent state takes place [Figs. 3(a) and 3(c)].[16] Correspondingly the power spectrum suffers a remarkable broadening. It should be noted that the peaks in the spectrum, which appear close to the eigenmode frequencies, are distributed also in the range of frequency in which the modes are stable. This suggests that energy flow from low-frequency unstable modes to high-frequency stable modes is induced by mode-mode coupling. In this regard the turbulent behavior in our system has resemblance to that seen in hydrodynamical systems[11] and in reaction-diffusion systems.[17]

Finally, let us briefly discuss the experimental possibility of observing the phenomena described above. In systems with large τ_R, most of the stationary solutions are unstable if $A^2B \gg 1$, as has been pointed out. For CS_2 placed in a Fabry-

Perot cavity of $L \sim l \sim 5$ cm and $R \sim 0.6$, for example, the minimum incident power required for the occurrence of a turbulent state is estimated to be 50 MW/cm^2, by using the values $\gamma^{-1} \sim 2$ psec, $n_2 \sim 10^{-11}$ esu and $k \sim 10^5$ cm^{-1}. This value of the incident power can be drastically lowered if a medium with large n_2 is used; 1 kW/cm^2 is sufficient for Rb vapor ($\gamma^{-1} \sim 200$ psec, $n_2 \sim 10^{-6}$ esu) in the same cavity.

An equivalent system whose dynamics obeys Eq. (8) can be constructed by modifying partly the hybrid bistable optical device studied by Garmire *et al.*[3] One has only to insert a delay line with delay time t_R between the photoconductor which detects the output light from a Pockels cell modulator and the feedback circuit. If the rise time of the detector is short enough, the equation which governs the temporal behavior of the output voltage is identical with Eq. (8), where γ^{-1} should be regarded as the relaxation time of the feedback circuit. In such a system the transition from self-pulsing to turbulence will easily be observed.

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Geometry from a Time Series

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It is shown how the existence of low-dimensional chaotic dynamical systems describing turbulent fluid flow might be determined experimentally. Techniques are outlined for reconstructing phase-space pictures from the observation of a single coordinate of any dissipative dynamical system, and for determining the dimensionality of the system's attractor. These techniques are applied to a well-known simple three-dimensional chaotic dynamical system.

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Lorenz originally demonstrated that very simple low-dimensional systems could display "chaotic" or "turbulent" behavior.[1] Attractors which display such behavior were termed "strange attractors" by Ruelle and Takens,[2] who then went on to conjecture that these strange attractors are the cause of turbulent behavior in fluid flow. The experiments of Gollub and Swinney have strengthened the conjecture,[3] but the question still remains: How can we discern the nature of the strange attractor underlying turbulence from observing the actual fluid flow?