Universality in Analytic Corrections to Scaling for Planar Ising Models

Amnon Aharony
Department of Physics, Harvard University, Cambridge, Massachusetts 02138, and Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel[a]

and

Michael E. Fisher
Baker Laboratory, Cornell University, Ithaca, New York 14853
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It is argued that the leading corrections to scaling for planar Ising models, which occur as analytic factors, arise from the quadratic terms of the nonlinear thermal and ordering fields (rather than from irrelevant variables). This yields $\alpha_{\rm eff}' + 2\beta_{\rm eff} + \gamma_{\rm eff}' = 2$ for the effective critical exponents, with *no* leading corrections, and is confirmed by exact square-lattice results for arbitrary anisotropy J_2/J_1. For isotropic lattices the ratios of correction amplitudes and quadratic nonlinear-field terms appear universal.

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As the critical point of a ferromagnet fluid, etc., is approached, the asymptotic behavior of a quantity, $L(T)$, as $T \to T_c^{\pm}$, can usually be characterized as

$$L(T) = L_0 \hat{t}^{\lambda}(1 + a_L^{\pm}\hat{t}^{\theta} + \ldots). \tag{1}$$

where L_0 is a constant and $\hat{t} = |T - T_c|/T$ [which variable proves more convenient here than the customary $t = (T - T_c)/T_c \equiv \pm\hat{t}/(1 \mp \hat{t})$]. The expression in parentheses in (1) represents the correction-to-scaling factor to the asymptotic power law with exponent λ. It has been evident for some time[1] that many real experiments, even when of the highest precision,[2] do not attain the truly asymptotic regime. Thus, one rather observes an ...ponent[1,2]

$$\lambda_{\rm eff} = \partial \ln L / \partial \ln \hat{t} = \lambda + \theta a_L^{\pm} \bar{t}^{\theta} + \ldots, \tag{2}$$

where $\bar{t}$ is a suitable average of $\hat{t}$ over the range of measurement.[3] The correction factor also plays an important role in analysis aimed at estimating λ accurately from series expansions for model systems.[4] Accordingly the nature and origin of the leading correction term, $a_L^{\pm}\hat{t}^{\theta}$, for quantities such as the free energy F, the spontaneous magnetization or order parameter M, and the susceptibility/compressibility χ (with exponents $2-\alpha$, β, and $-\gamma$) are matters of continuing significance to theory and experiment.

In this note we discuss the issue for planar Ising models, where many exact theoretical results are available,[5,6] although the literature discussing the corrections is somewhat confusing. Usually, the corrections to scaling arise, in a renormalization-group viewpoint,[7,8] from the leading irrelevant variables. We argue, however, that in the planar Ising models they arise, instead, from the *nonlinear scaling fields*.[7,8] Furthermore, we uncover a novel universal relation satisfied by the nonlinear fields for different lattices and show that the effective-exponent relation

$$\alpha_{\rm eff}' + 2\beta_{\rm eff} + \gamma_{\rm eff}' = 2 \tag{3}$$

holds widely *without* any leading corrections.[9] This follows from a general relation between the correction amplitudes a_M, a_F, and a_χ [Eq. (10) below]. The scaling law (3) has also been proved recently[10] to leading order in $\epsilon = 4 - d$ (together with[9] $\alpha_{\rm eff}' = \alpha_{\rm eff}$ and $\gamma_{\rm eff}' = \gamma_{\rm eff}$), and seems to agree reasonably with experiments and with series-expansion evidence for systems of dimensionality $d = 3$ (even though it is broken analytically in higher order[11]). For the $d = 2$ Ising models, however, we find $\alpha_{\rm eff}' = -\alpha_{\rm eff}$ and $(\gamma_{\rm eff}' - \frac{7}{4}) = -(\gamma_{\rm eff} - \frac{7}{4})$ (without leading corrections). For the isotropic models we find additional universal ratios, e.g., $(\gamma_{\rm eff}' - \frac{7}{4})/(\beta_{\rm eff} - \frac{1}{8}) = -\frac{2}{9}$ [see Eq. (4) below.] Our various results invite experimental tests and further theoretical investigation in both Ising-like and other types of two-dimensional system. Likewise the effects of nonlinear scaling fields and the possibility of associated universal relations should also be explored theoretically and experimentally in three-dimensional systems.

To start, recall that the formal machinery of renormalization-group analysis[7] shows that important corrections to asymptotic power-law behavior can arise from "irrelevant variables," that is, physical parameters beyond the one, two, or a few "relevant variables" whose variation destroys criticality (or changes the exponents). In particular for small $\epsilon = 4 - d$, the leading corrections in (1) are expected to occur as confluent *singularities* with θ nonintegral and less than unity. Specifically, if the leading irrelevant variable, say w, scales under renormalization as b^{λ_w}, where b is the corresponding rescaling factor for lengths, then, in leading order, the free energy should be a function only of the scaled combination $z = w/t^{\lambda_w/\lambda_t} = w\hat{t}^{\omega\nu}$. Here $\nu = 1/\lambda_t$ is the standard correlation-length exponent and $\omega = -\lambda_w > 0$. Normally an expansion of the free-energy scaling function in integral powers of z is permissible[7]: In (1) this yields the identifications (i) $\theta = \omega\nu$ and (ii) $a_L^{\pm} = b_L^{\pm} w$, where the coefficients $b_L^{\pm}$ depend only on the scaling function which (being determined by the fixed-point vicinity) is universal up to "metrical" or scale factors. Thus, *ratios* such as $a_F^{\pm}/a_M$ and $a_\chi^{\pm}/a_M$ should also be universal.[10]

A crucial feature in extending this line of argument to planar Ising models, however, is the fact that all correction-to-scaling factors are then known to be analytic in T through T_c.[5,6] Recall also that $\nu = 1$, $\alpha = 0$, $\beta = \frac{1}{8}$, $\gamma = \frac{7}{4}$ but that $\hat{t}^{-\alpha}$ must be replaced by $-\ln\hat{t}$. The analyticity implies (i) $\theta \equiv 1$ and (ii) $a_L^- = -a_L^+$ (implying, e.g., $\alpha_{\rm eff}' = -\alpha_{\rm eff}$). Now, using the exact results for the spatially isotropic triangular, square, honeycomb, and kagome lattices, one can compute[12] the $a_L^{\pm}$: Thence we find

$$a_F^{\pm}/a_M = \mp\tfrac{16}{9} \text{ and } a_\chi^{\pm}/a_M = \mp\tfrac{2}{9}, \tag{4}$$

for all four isotropic lattices—apparently a confirmation of the anticipated universality.

There have been various attempts recently[6b,12,13] to account for universal features of the planar-Ising-model correction amplitudes $a_L^{\pm}$. The first postulates were too restrictive and failed for the kagome lattice. Later Gaunt and Guttmann[12b] allowed themselves a third metrical factor (basically equivalent to allowing for an irrelevant variable w) and resolved the difficulty. Rephrased, we believe more transparently, the *only combinations* of the $a_L^{\pm}$ which Gaunt and Guttmann find to be universal are simple ratios such as those in (4). Then knowledge of any one a_L^+ or a_L^- for a given lattice uniquely determines all the others.

In view of the result $\theta = 1$, it is certainly tempting to associate the leading corrections in the planar Ising model with some irrelevant variable, w, scaling with an exponent $\omega = 1$ and, hence, in the same way as an inverse length. The only obvious candidate for such a variable is the *reciprocal-lattice spacing,* a^{-1} (related to the momentum cutoff in continuum-space versions of Ising models). However, the exact solutions for the thermodynamics of planar Ising models[5] exhibit no such explicit dependence on a![14] This fact, combined with the *analyticity* of the correction factors, leads us to focus on a *second source of corrections* to pure power laws within renormalization-group theory: Explicitly, the full nonlinear recursion relations, like

$$\pm\hat{t}' = \pm b^{\lambda_t}\hat{t} + R_2(b)\hat{t}^2 + R_3(b)h^2 + \dots, \tag{5}$$

where h is the ordering field, can be replaced by purely linear relations, provided that suitable *nonlinear scaling fields* are introduced.[7,8] Formally one may construct the thermal and ordering nonlinear fields as analytic functions,

$$g_t = \pm\hat{t}[1 \pm c_t\hat{t} + O(\hat{t}^2, h^2)], \tag{6}$$

$$g_h = h[1 \pm c_h\hat{t} + O(\hat{t}^2, h^2)], \tag{7}$$

such that the linear recursion relations $g_t' = b^{\lambda_t} g_t$ and $g_h' = b^{\lambda_h} g_h$ are satisfied *without further corrections*. [Note that in writing (6) and (7), asymptotic symmetry under $h \to -h$ has been assumed.] The singular part of the free energy should then scale as

$$F_s(T,h) \approx |g_t|^{2-\alpha} Y_{\pm}(g_h/|g_t|^{\Delta}), \tag{8}$$

with $\Delta = \beta + \gamma$, and where now we neglect any irrelevant variables (which, however, should in principle also enter the nonlinear scaling fields). On substituting (6) and (7) and differentiating as required, we recapture (1) but now with the identifications (i) $\theta \equiv 1$ and (ii)

$$\begin{aligned} a_F^{\pm} &= \pm(2-\alpha)c_t, \\ a_M &= -c_h - \beta c_t, \\ a_\chi^{\pm} &= \pm(2c_h - \gamma c_t). \end{aligned} \tag{9}$$

Notice first that these expressions satisfy

$$a_F^- - 2a_M + a_\chi^- \equiv 0, \tag{10}$$

from which the effective-exponent relation (3) follows (to leading order) immediately and quite generally! Second, specialize to the planar isotropic Ising models and use *either* member of (4) to match (9): One discovers that *both* lead to the *identical result relating the two nonlinear scaling fields*, namely,

$$\mathcal{R}(1) = c_t/c_h = 1, \tag{11}$$

which thus appears as a *universal ratio over the isotropic planar lattices.* Consequently, all the leading correction-to-scaling amplitude ratios for the spatially isotropic planar Ising lattices seem to be dictated by a single universal ratio of the quadratic terms in the nonlinear scaling fields.

To check the generality of these surprising results, consider the anisotropic square lattice with coupling constants J_1 and $J_2 = \kappa J_1$ along the x and y axes. One can then obtain[5,15]

$$a_F^{\pm} = \pm \tau_2(1+\kappa^2\sigma_1^2)K_1^c/(1+\kappa\sigma_1), \tag{12}$$

$$a_M = [1 - 3\tau_2^2 - 4\kappa\tau_1\tau_2 + \kappa^2(1-3\tau_1^2)]K_1^c/\,8\tau_2(1+\kappa\sigma_1), \tag{13}$$

where $K_i^c = J_i/k_B T_c$, $\sigma_i = \sinh 2K_i^c$, and $\tau_i = \cosh 2K_i^c$ $(i=1,2)$. An explicit expression for a_χ^+ in terms of $z_i^c = \tanh K_i^c$ is given in Ref. 6a, Eq. (16). At the cost of some algebra which relies heavily on the in-criticality conditions $\sigma_1\sigma_2 = 1 = z_1^c + z_2^c + z_1^c z_2^c$, one can now verify that the relation (10), and thence (3), holds as an identity for *all anisotropy* κ—a remarkable results which strongly supports the contention that the leading correction terms in the planar Ising models arise solely from the nonlinear scaling fields.

One discovers, however, that the ratio $\mathcal{R}(\kappa) = c_t/c_h$ does have a nontrivial dependence on κ even though $\mathcal{R}(1)=1$ for all four standard lattices; likewise the ratios in (4) depend on κ. For the *anisotropic* case we thus conclude that the ratios (4), and hence ratios like $(\gamma_{\rm eff}' - \frac{7}{4})/(\beta_{\rm eff} - \frac{1}{8})$, have explicit dependence on κ (and therefore are nonuniversal!). Actually this dependence on κ should not be so surpising since *spatial anisotropy* is believed to be a *marginal operator* in the renormalization-group sense.[16] Furthermore, one knows[5] that the anisotropy of the planar–Ising-model correlation decay, *even at* T_c, depends explicitly on κ. One may conclude that a *line of fixed points* describes the variation of certain features, like the nonlinear scaling fields with κ, even though the exponents and thermodynamic scaling functions remain invariant.

Finally, a comment on $d=3$. Clearly, different origins for the leading correction terms in $d=2$ and for $d \lesssim 4$ preclude any simple interpolation of ratios such as a_F^+/a_χ^+ between the two limits. In $d=3$, where most evidence indicates[2,7] $\omega\nu \simeq 0.5$, the mechanism of the irrelevant variable is probably the most important source of corrections to scaling. However, the nonlinear scaling fields and the resulting analytic corrections must still play a role: This could be especially significant if w is relatively small (as seems to be indicated by series-expansion studies[4] for general spin-S Ising models, which suggest that w is close to zero for $S=\frac{1}{2}$). The competition between both types of correction terms might then be important in a definitive elucidation of the series expansions.[12b,17].

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[(a)]Permanant address.

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Cross-Section Measurements for Charm Production by 209-GeV Muons

A. R. Clark, K. J. Johnson, L. T. Kerth, S. C. Loken, T. W. Markiewicz, P. D. Meyers, W. H. Smith, M. Strovink, and W. A. Wenzel
Physics Department and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720

and

R. P. Johnson, C. Moore, M. Mugge, and R. E. Shafer
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

and

G. D. Gollin,[a] F. C. Shoemaker, and P. Surko[b]
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 10 April 1980)

Interactions of 209-GeV muons in the multimuon spectrometer at Fermilab have yielded 20 072 dimuon final states, with (81 ± 10)% attributed to production of charmed states decaying to muons. The cross section for diffractive charm muoproduction is $6.9^{+1.9}_{-1.4}$ nb. Extrapolated to $Q^2 = 0$, the effective cross section for 178- (100-) GeV photons is 750^{+180}_{-130} (560^{+200}_{-120}) nb.

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Real- and virtual-photon beams are able to elucidate charm production in hadron reactions because they substitute charge for color coupling at one vertex. Charm and forward-ψ photoproduction rates limit the ψN total cross section without assuming vector-meson dominance (VMD), and within VMD yield the ratio of elastic to inelastic ψN scattering.[1] Charm muoproduction data directly test the photon-gluon-fusion (γGF) model,[2] which uses elements of quantum chromodynamics. This Letter presents charm-production cross sections which impose significant model constraints. Differential spectra appear in a second paper.[3]

One model-dependent measurement of the charm-muoproduction cross section at 270 GeV has been reported[4] as 3 ± 1 nb. Wide-band photon-beam experiments have measured cross sections for inclusive D^0 production averaged from 50 to 200 GeV of 464 ± 207 nb[5] and 295 ± 130 nb.[6] In no case has discrimination between charm-production models been attempted.

This experiment identifies charmed states by their n-body ($n \geqslant 3$) decays into muons. Unresolved charmed hadrons contribute in proportion to their production rate and leptonic branching ratio. While unsuited to a first observation of charmed states, this continuum charm signature is the only reasonable explanation for (81 ± 10)% of the 20 072 single-extra-muon events reported here. These high statistics, coupled with full determination of virtual-photon four-momenta, permit the study of charm-production mechanisms.

The spectrometer has been described earlier.[7] The $\geqslant 2\mu$ trigger required a $\gtrsim$ 20-GeV hadronic shower $\gtrsim$ 2 m upstream of $\geqslant$ 2 hits in each of three consecutive trigger hodoscopes. Full tracking capability in an area including the beam produced a high, nearly Q^2-independent acceptance. Data are reported from 1.4×10^{11} positive and 0.3×10^{11} negative Fermilab beam muons at 209 GeV. For $\mu^+\mu^+$ or $\mu^-\mu^-$ final states, the scattered muon is chosen to be the more energetic muon. This algorithm is 91% successful when checked using $\mu^+\mu^-$ events. Regions of rapidly varying acceptance are excluded by requiring daughter muon energies to exceed 15 GeV, vertices to lie in the upstream 60% of the target, and shower energies to exceed 36 GeV. Muon trident contamination is reduced by requiring the daughter muon to possess $\geqslant$ 0.45 GeV/c momentum transverse