PHYSICAL REVIEW LETTERS

VOLUME 45 | 1 SEPTEMBER 1980 | NUMBER 9

Nonlinear Partial Difference Equations for the Two-Dimensional Ising Model

Barry M. McCoy

The Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794

and

Tai Tsun Wu

Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

(Received 30 April 1980)

The two-point function of the two-dimensional Ising model at arbitrary temperature is expressed in terms of the solution of a nonlinear partial difference equation. From this difference equation the known results for the two-point function of the Ising field theory may be regained as a special case.

PACS numbers: 05.50.+q

In the past several years much exact information about spin-correlation functions of the two-dimensional Ising model has been obtained. In particular, dispersion expansions of the two-spin[1] and more generally of n-spin correlations[2] have been obtained. In the scaling limit the two-spin[1,3,4] correlation has been expressed in terms of the solution of the sinh-Gordon equation. Furthermore, Sato, Miwa, and Jimbo[5] showed that nonlinear partial differential equations can be found for the n-point functions in the scaling limit. Because of the field theoretic interpretation of this scaling limit[2,6] these nonlinear equations are of great importance.

It is the purpose of the present note to point out that the connection between the correlation functions and closed systems of equations occurs on an even more fundamental level. At arbitrary temperatures, not necessarily in the scaling limit, the n-point correlation functions can be expressed in terms of *nonlinear partial difference equations*. The differential equations mentioned above can be reproduced by taking the scaling limit.

These difference equations have several important consequences.

(1) From the difference equations we may determine the boundary conditions on the scaling-limit differential equations. These boundary conditions connect the scaling behavior to the underlying lattice and are an expression of the renormalization procedure used to construct the field theory.

(2) These difference equations may be used to study questions such as corrections to the $T=T_c$ large-separation behavior of the correlations, and the value of the susceptibility of the antiferromagnet at T_c.

While the procedure of dealing with the n-spin correlation function is elementary and is basically the same for all n, there are several technical complications for $n>2$, such as the doubling of the number of dependent variables, that make it difficult to present in a concise and clear manner. Therefore, we shall confine our attention here to the two-point function. These present considerations may be viewed as a generalization of the procedure of Ref. 4, not Ref. 3 or 5.

Let E_1 (E_2) be the horizontal (vertical) interaction energies, let N (M) be the horizontal (vertical) coordinates, define[7]

$$z_i = \tanh E_i/kT, \quad i = 1, 2, \tag{1}$$

$$\Delta(x, y) = a - \tfrac{1}{2}\gamma_1(x + x^{-1}) - \tfrac{1}{2}\gamma_2(y + y^{-1}), \tag{2}$$

with

$$a = (1 + z_1^2)(1 + z_2^2), \quad \gamma_1 = 2z_2(1 - z_1^2), \quad \gamma_2 = 2z_1(1 - z_2^2), \tag{3}$$

and let[8]

$$\mathfrak{M} = |1 - [\sinh(2E_1/kT)\sinh(2E_2/kT)]^{-2}|^{1/8}. \tag{4}$$

Further, we follow (3.66) and (3.99) of Ref. 1 and (15) of Ref. 3 to define

$$f_{MN}(\lambda; \{\epsilon\}) = 1 + \sum_{n=1}^{\infty} \lambda^{2n} f_{MN}^{(2n)}(\{\epsilon\}), \tag{5}$$

with

$$f_{MN}^{(2n)}(\{\epsilon\}) = (-1)^{n+1}\gamma_1^{2n}\int \prod_{l=1}^{2n}\left(\frac{dx_l}{2\pi i x_l}\,\frac{dy_l}{2\pi i y_l}\,\Delta^{-1}(x_l, y_l)\right) x_1^{-M} x_2^{M} \cdots x_{2n}^{M} \times y_1^{-N} y_2^{N} \cdots y_{2n}^{N} x_1 \prod_{l=1}^{2n-1}\left(\frac{x_l^{-1} - x_{l+1}}{1 - y_l y_{l+1}^{-1} + \epsilon_l}\right), \tag{6}$$

and define

$$G(M, N; \lambda; \{\epsilon\}) = \sum_{k=0}^{\infty} \lambda^{2k+1} G^{(2k+1)}(M, N; \{\epsilon\}), \tag{7}$$

with

$$G^{(2k+1)}(M, N; \{\epsilon\}) = (-1)^k \gamma_1^{2k+1/2}\gamma_2^{1/2}\int \prod_{l=1}^{2k+1}\left(\frac{dx_l}{2\pi i x_l}\,\frac{dy_l}{2\pi i y_l}\,\Delta^{-1}(x_l, y_l)\right) \times x_1^{-M} x_2^{M} \cdots x_{2k+1}^{-M} y_1^{-N} \cdots y_{2k+1}^{-N} \prod_{l=1}^{2k}\left(\frac{x_l^{-1} - x_{l+1}}{1 - y_l y_{l+1}^{-1} + \epsilon_l}\right). \tag{8}$$

Here the integration contours are $|x_l| = |y_l| = 1$ and each ϵ_l either $\to 0+$ or $0-$. Then, when $M \neq 0$, we have for $T < T_c$

$$\langle \sigma_{00}\,\sigma_{MN}\rangle = \mathfrak{M}^2 \exp - \sum_{k=N}^{\infty} \ln f_{Mk}(1; \{\epsilon\}), \tag{9a}$$

and for $T > T_c$

$$\langle \sigma_{00}\sigma_{MN}\rangle = \mathfrak{M}^2 G(M, N; 1; \{\epsilon\}) \exp - \sum_{k=N}^{\infty} \ln f_{Mk}(1; \{\epsilon\}). \tag{9b}$$

In (6) and (8), if $M \neq 0$, we may carry out the x_l integrals by closing on the zeroes of Δ and find that there are in fact no poles in y_l. Thus, when $M \neq 0$, (9) is valid for all sets $\{\epsilon\}$. When $M = 0$, however, there are additional contributions from $x_l = 0$ and ∞ which do lead to poles in y_l. These extra terms must be canceled if (9) is to hold for $M = 0$. This may be efficiently carried out if we define

$$G^{\pm}(M, N; \lambda) = G(M, N; \lambda; \epsilon_{2l} = \pm\epsilon, \epsilon_{2l+1} = \mp\epsilon) \tag{10a}$$

and

$$f_{MN}^{\pm}(\lambda) = f_{MN}(\lambda; \epsilon_{2l} = \pm\epsilon, \epsilon_{2l+1} = \mp\epsilon) \tag{10b}$$

with $\epsilon \to 0^+$. Then (9) holds for $M = 0$ and $N > 0$ if we use G^+ and f_{MN}^+ and for $M = 0$ and $N < 0$ if we use G^- and f_{MN}^-. We also have

$$G^{+}(M, N; \lambda) = G^{-}(M, N; \lambda) \text{ if } M \neq 0, \tag{11a}$$

and

$$G^+(0, 0; \lambda) = G^-(0, 0; \lambda). \tag{11b}$$

Because of (11), it is useful to define

$$G(M, N; \lambda) = \begin{cases} G^+(M, N; \lambda) \text{ for all } M \text{ and } N \text{ except } M=0 \text{ and } N<0, \\ G^-(M, N; \lambda) \text{ for all } M \text{ and } N \text{ except } M=0 \text{ and } N>0, \end{cases} \tag{12}$$

and similarly for $f_{MN}(\lambda)$.

We proceed to indicate the derivation of the partial difference equation for $G(M, N; \lambda)$. The following two elementary algebraic relations are needed:

$$\gamma_1(1-x_1x_2^{-1})(x_1^{-1}-x_2) = -\gamma_2(1-y_1y_2^{-1})(y_1^{-1}-y_2) + \Delta(x_1, y_1) - \Delta(x_2, y_2) \tag{13}$$

and

$$\prod_{l=0}^{k} x_{2l+1} \prod_{j=1}^{k} x_{2j}^{-1} + \prod_{l=0}^{k} x_{2l+1}^{-1} \prod_{j=1}^{k} x_{2j} - \sum_{l=0}^{k}(x_{2l+1}+x_{2l+1}^{-1}) + \sum_{l=1}^{k}(x_{2l}+x_{2l}^{-1})$$
$$= -\sum_{j=1}^{k}\sum_{l=j}^{k}(1-x_{2j-1}x_{2j}^{-1})(1-x_{2l}x_{2l+1}^{-1})\left(\prod_{i=j}^{l} x_{2i+1}\prod_{m=j}^{l} x_{2m}^{-1} + \prod_{i=j-1}^{l-1} x_{2i+1}^{-1}\prod_{m=j}^{l-1} x_{2m}\right). \tag{14}$$

In view of the appearance of γ_1 and γ_2 in (2), the useful lattice Laplacian is

$$\nabla_L^2\Phi(M,N) = \gamma_1[\Phi(M+1,N)+\Phi(M-1,N)-2\Phi(M,N)] + \gamma_2[\Phi(M,N+1)+\Phi(M,N-1)-2\Phi(M,N)]. \tag{15}$$

From (13) and (14), the application of the lattice Helmholtz operator to G gives, after extensive rearrangement,

$$[\nabla_L^2 - 2(a-\gamma_1-\gamma_2)]\, G(M,N;\lambda) = -\gamma_2 L(M,N;\lambda)[G(M+1,N;\lambda)] + G(M-1,N;\lambda)] + M \leftrightarrow N \tag{16}$$

for all M and N except $M=N=0$. In the exchange $M \leftrightarrow N$ we always include $\gamma_1 \leftrightarrow \gamma_2$. Here L is given by

$$L(M,N;\lambda) = H(M,N;\lambda)\bar{H}(M,N;\lambda) - G^2(M,N;\lambda), \tag{17}$$

where H and $\bar{H}$ are defined in the same way as G except that there are extra factors of y_{2k+1}^{-1} and y_1, respectively, in the integrand. Repeated application of (13) then leads to the recurrence relation

$$\begin{pmatrix} 1 & \gamma_1\gamma_2^{-1}G(M,N+1;\lambda)G(M,N-1;\lambda) \\ \gamma_1^{-1}\gamma_2 G(M+1,N;\lambda)G(M-1,N;\lambda) & 1 \end{pmatrix}\begin{pmatrix} L(M,N;\lambda) \\ \tilde{L}(M,N;\lambda)\end{pmatrix}$$
$$= \begin{bmatrix} G^2(M,N;\lambda) - G(M,N+1;\lambda)G(M,N-1;\lambda) \\ G^2(M,N;\lambda) - G(M+1,N,\lambda)G(M-1,N;\lambda)\end{bmatrix}, \tag{18}$$

where $\tilde{L}$ is obtained from L by $M \leftrightarrow N$.

It is now straightforward to solve (18) and substitute into (17) to find for all M and N except $M=N=0$ where there is a source

$$[\nabla_L^2 - 2(a-\gamma_1-\gamma_2)]\,G(M,N;\lambda)$$
$$= -[1 - G(M+1,N;\lambda)G(M-1,N;\lambda)G(M,N+1;\lambda)G(M,N-1;\lambda)]^{-1}[G(M+1,N;\lambda)+G(M-1,N;\lambda)]$$
$$\times\{\gamma_2[G^2(M,N;\lambda) - G(M,N+1;\lambda)G(M,N-1;\lambda)]$$
$$-\gamma_1 G(M,N-1;\lambda)G(M,N+1;\lambda)[G^2(M,N;\lambda) - G(M+1,N;\lambda)G(M-1,N;\lambda)]\} + M \leftrightarrow N. \tag{19}$$

This is the desired partial difference equation for G.

A more symmetrical form of (19) is

$$2aG(M,N;\lambda)[1 - G(M+1,N;\lambda)G(M-1,N;\lambda)G(M,N+1;\lambda)G(M,N-1;\lambda)]$$
$$= [\gamma_1 + \gamma_2 G^2(M,N;\lambda)]\{G(M+1,N;\lambda) + G(M-1,N;\lambda) - [G(M,N+1;\lambda) + G(M,N-1;\lambda)]$$
$$\times G(M+1,N;\lambda)G(M-1,N;\lambda)\} + M \leftrightarrow N. \tag{20}$$

Thus G and G^{-1} satisfy the same equation.

The partial difference equation satisfied by

$$\eta(M, N; \lambda) = [1 - G(M, N; \lambda)][1 + G(M, N; \lambda)] \tag{21}$$

is

$$[(a-\gamma_1-\gamma_2)-(a+\gamma_1+\gamma_2)\eta^2][\eta^+ + \eta^- + \eta_+ + \eta_-] + [(a+\gamma_1+\gamma_2)-(a-\gamma_1-\gamma_2)\eta^2][\eta^-\eta_+\eta_- + \eta^+\eta_+\eta_- + \eta^+\eta^-\eta_- + \eta^+\eta^-\eta_+] + 2(\gamma_1-\gamma_2)\eta(\eta^+\eta^- - \eta_+\eta_-) = 0\,, \tag{22}$$

where

$$\eta^\pm = \eta(M \pm 1, N; \lambda), \quad \eta_\pm = \eta(M, N \pm 1; \lambda)\,. \tag{23}$$

In the scaling limit[1,3-5] (22) simplifies to a Painlevé equation. Thus (22) may be called a lattice Painlevé equation. The exponential change of variable $\eta = e^{-\psi}$ then gives the lattice sinh-Gordon equation satisfied by ψ.

Finally we must relate $f_{MN}(\lambda)$ to $G(M, N; \lambda)$. This is easily obtained by calculating a first difference

$$f_{M+1,N}(\lambda) - f_{M,N}(\lambda) = G(M, N; \lambda)\, G(M+1, N+1; \lambda) f_{M+1,N}(\lambda) - G(M+1, N; \lambda)\, G(M, N+1; \lambda) f_{M,N}(\lambda)\,. \tag{24}$$

This work has been supported in part by the National Science Foundation under Grant No. DMR-79-08556 and by the United States Department of Energy under Contract No. DE-AS02-76-ER03227.

[1]T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Phys. Rev. B 13, 316 (1976).

[2]B. M. McCoy, C. A. Tracy, and T. T. Wu, Phys. Rev. Lett. 38, 793 (1977); R. F. Bariev, Physica (Utrecht) 83A, 388 (1976), and 93A, 137 (1978), and Phys. Lett. 64A, 164 (1977), and 68A, 175 (1978); D. B. Abraham, Phys. Lett. 61A, 271 (1977).

[3]E. Barouch, B. M. McCoy, and T. T. Wu, Phys. Rev. Lett. 31, 1409 (1973); C. A. Tracy and B. M. McCoy, Phys. Rev. Lett. 31, 1500 (1973).

[4]B. M. McCoy, C. A. Tracy, and T. T. Wu, J. Math. Phys. 18, 1058 (1977).

[5]M. Sato, T. Miwa, and M. Jimbo, Proc. Jpn. Acad. 53A, 147, 153, and 183 (1977).

[6]For a review discussion of this connection, see, for example, B. M. McCoy and T. T. Wu, in *Bifurcation Phenomena in Mathematical Physics and Related Topics*, edited by C. Bardos and D. Bessis (D. Reidel Publishing Co., Hingham, Mass., 1980), p. 69.

[7]L. Onsager, Phys. Rev. 65, 117 (1944).

[8]C. N. Yang, Phys. Rev. 85, 808 (1952).