Self-Consistent Model of Stochastic Magnetic Fields

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In this Letter, a model is described in which spatially stochastic magnetic field and current fluctuations in a plasma with sheared average magnetic field are treated consistently with Ampere's Law.

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Recently, a number of calculations have been published which deal with the problem of stochastic magnetic field lines in a plasma with magnetic shear. In the simplest version of the problem, the field is constant in time but the lines diffuse in x and y as one moves in the z direction. For the most part, recent work has concentrated on the "stochastic acceleration" problem in which the statistical properties of the field line trajectories are determined, given the perturbed field or current density. The problem of the correlated diffusion of two field lines has also been investigated.[1,2] However, there has not been published a calculation in which the field stochasticity, including two-line correlation, is made consistent with an average z-independent current profile and Ampere's Law. In this Letter, we present such a calculation. We describe a self-consistent case of finite-β drift waves, but only a single-line calculation is done.[3] In order to illustrate our procedure, we consider a concrete example—nonlinear tearing modes. To keep the example as simple as possible, we have used a rather idealized model of tearing-mode turbulence. However, we believe the concepts and techniques are useful in a much broader context.

Rutherford[4] has argued that when the width of the magnetic island exceeds the tearing-layer width, the tearing mode enters a nonlinear phase described (in a sheet pinch) by

$$\vec{B}\cdot\nabla J=0, \tag{1a}$$

$$\nabla_\perp^2\psi=4\pi J, \tag{1b}$$

$$\partial\psi/\partial t+\vec{v}_\perp\cdot\nabla\psi=\eta J-\partial\varphi/\partial z. \tag{1c}$$

Equation (1a) states that the z component of the current, J, is constant along the field lines $\vec{B}$. If we note that ψ is a flux function such that $\vec{B}=B_0\hat{z}+\hat{z}\times\nabla\psi$, (1b) is recognized as Ampere's Law ($\perp$ denotes quantities perpendicular to $\hat{z}$). If the resistivity, η, is zero, (1c) states that the flux, ψ, is frozen into the plasma fluid of flow velocity $\vec{v}_\perp=B_0^{-1}\nabla\varphi\times\hat{z}$. With $\vec{E}=\nabla\varphi$ as the electric field, Eq. (1c) follows from Ohm's law and Faraday's law. White *et al.*[5] have also considered a model based on Eq. (1).

We decompose the total current fluctuation δJ into two constituents, $J^{(c)}$ and $\tilde{J}$. $J_{k_y,k_z}{}^{(c)}$ is the coherent current fluctuation produced in response to a magnetic perturbation of wave numbers k_z and k_y, and which has the same phase as that perturbation. $J_{k_y,k_z}{}^{(c)}$ is calculated using a renormalized version of (1a). $\tilde{J}$ denotes the incoherent current fluctuations due to the highly nonlinear rearrangement of the average-current profile $\langle J\rangle_{\rm av}$ at the $\vec{k}\cdot\vec{B}=0$ resonance.[6] Phenomena which could be interpreted in terms of this rearrangement process have been observed in computational studies of the nonlinear evolution of the tearing mode.[7]

Because $\tilde{J}$ is a very complicated function, the theory deals only with the correlation function $\langle\tilde{J}(1)\tilde{J}(2)\rangle$ (1 means x_1,y_1,z_1) which we compute in the "stochastic acceleration" approximation using (1a). Next we use (1b) to require that the current $J^{(c)}$ and $\tilde{J}$ are consistent with the magnetic fields assumed in their calculation. This procedure is analogous to that of the clump theory of Vlasov phase-space density fluctuations.[6]

To determine the coherent current response, note that (in a sheet pinch) (1a) may be written as

$$\left(\frac{\partial}{\partial z}+\frac{x}{L}\frac{\partial}{\partial y}\right)J+b\frac{\partial}{\partial x}J=-b\frac{\partial}{\partial x}\langle J\rangle_{\rm av}. \tag{2}$$

Here, $\partial\langle J\rangle_{\rm av}/\partial x$ is the average-current gradient, and $b=B_x/B_0$. We have put $B_y=+xB_0/L$, where L is the shear length. Also, since the nonlinear term $B_y\partial J/\partial y$ is unimportant in (2), we have retained only the x component of the magnetic field perturbation.

A solution to (2) may be obtained by noting that its characteristic equations determine the trajectory of a magnetic field line. Using z to parametrize progression along a field line, the turbulent line trajectories are given by $dx/dz=b$ and $dy/dz=[x+\delta x(z)]/L$, where $\delta x(z)=\int_0^z dz'\,dx(z')/dz'$. We integrate along field line trajectories from $z' = -\infty$ to z. We assume the field lines diffuse in x as one moves in the positive z direction. This ensures the convergence of the trajectory integral and permits us to neglect the contribution from $J^{(c)}(-\infty)$. As will become evident later, if the turbulence is homogeneous in z, we would obtain the same final answer by integrating from $+\infty$ to z. If we carry out the trajectory integrals, Fourier transform the y and z dependence, and make a cumulant[8] expansion, we obtain

$$J_{k_y,k_z}^{(c)} = \int_0^\infty dz \exp\left\{iz\left(k_z - \frac{k_y x}{L}\right) - \frac{k_y^2}{2L^2}\langle[\int_0^z dz'\,\delta x(z')]^2\rangle_{\rm av}\right\} b_{k_y,k_z}\frac{\partial\langle J\rangle_{\rm av}}{\partial x}. \quad (3)$$

Finally, noting that $\langle\delta x^2\rangle_{\rm av} = 2Dz$, where D is the turbulent magnetic diffusion coefficient (in the x direction) defined below, the renormalized coherent current response is

$$J_{k_y,k_z}^{(c)} = -R(k_z - k_y x/L) b_{k_y,k_z}\partial\langle J\rangle_{\rm av}/\partial x, \quad (4)$$

where

$$R(s) = \int_0^\infty dz \exp(isz - k_y^2 D z^3/3L^2), \quad (5)$$

$$D = \int (dk_y/2\pi)\int(dk_z/2\pi)\langle b^2\rangle_{k_y,k_z} R(k_z - k_y x/L). \quad (6)$$

Here, $\langle b^2\rangle_{k_y,k_z}$ is the Fourier transform (in $y_1 - y_2$ and $z_1 - z_2$) of the correlation function, $\langle b(1)b(2)\rangle$, for the magnetic field perturbation.[6] The $\vec{k}\cdot\vec{B} = 0$ singularity when $D = 0$ in (4) is characteristic of the outside solution for the tearing mode,[9] and is removed here by the field-line diffusion coefficient.

$z_t = (k_y^2 D/3L^2)^{-1/3}$ is the basic single-line z randomization length in the theory.[3,10] A field line and a magnetic perturbation are in resonance when $|k_z - k_y x/L| \le z_t^{-1}$. Therefore, the resonance width in x is given by $\Delta x_t = (LD/3k_y)^{1/3}$. Writing Δx_t in terms of the resonant portion of the spectrum only ($\langle b^2\rangle_{\rm res}$) and setting $R \sim z_t$, we find $D \sim \langle b^2\rangle_{\rm res} z_t$. If we use this expression for D, it is easily verified that $\Delta x_t \sim [\langle b^2\rangle_{\rm res} L^2/k_y^2]^{1/4}$—an expression identical with the width of the separatrix of a single magnetic island.[11] Δx_t can be used to ascertain the amplitude necessary for island overlap,[12] and, therefore, the onset of turbulence.

The coherent current response given by (4) can be substituted into Ampere's Law (1b) to obtain an equation for ψ:

$$(\partial^2/\partial x^2 + k_y^2)\psi_{k_y,k_z}(x) + [4\pi k_y B_0^{-1}(\partial\langle J\rangle_{\rm av}/\partial x)/(k_z - k_y x/L + iz_t^{-1})]\psi_{k_y,k_z}(x) = 4\pi\tilde{J}_{k_y,k_z}(x), \quad (7)$$

where $R(k_z - k_y x/L)$ has been approximated by $i(k_z - k_y x/L + iz_t^{-1})^{-1}$. In the limit of $z_t \to \infty$ and $\tilde{J} = 0$, (7) is recognizable as the outside equation of the linear tearing-mode theory.[9]

In order to solve (7) for $\tilde{J} \ne 0$, it is useful to define a Green's function $G(x,x')_{k_y,k_z}$ which satisfies (7) with the right-hand side set equal to $\delta(x - x')$, and with the same boundary conditions. It turns out that we will only need the Green's function evaluated at $x = x' = x_s$ where $x_s = k_z L/k_y$ is the location of the resonant surface. $G(x,x')_{k_y,k_z}$ has a simple form when $z_t^{-1} \to 0^+$, and can be obtained by integrating (7) from $x = -\infty$ to $x = +\infty$ and using the Plemelj formulas. We get

$$G(x = x_s, x' = x_s)_{k_y,k_z} = [\Delta'_{k_y,k_z} + i4\pi^2 L k_y(\partial\langle J\rangle_{\rm av}/\partial x)/B_0|k_y|]^{-1}. \quad (8)$$

Δ' is the usual tearing-mode stability parameter[9] equal to the discontinuity in the logarithmic derivative of the real part of the homogeneous solution to (7), i.e., with the right-hand side set equal to zero. The imaginary part in braces in (8) comes from the discontinuity in the logarithmic derivative of the imaginary part of the solution (for $z_t^{-1} \to 0^+$). This can be seen by noting that near the singularity the homogeneous solution of (7) has the form $\psi(x) = \psi(w_s)(w - w_s)\ln(w - w_s) +$ regular function, where $w = 4\pi L B_0^{-1} x\,\partial\langle J\rangle_{\rm av}/\partial x$.

A bivariate diffusion equation for the fluctuation correlation function $\langle\delta J(1)\delta J(2)\rangle$ can be obtained by the procedures used in Ref. 6. We define the variables $x_\pm = x_1 \pm x_2$ and $y_\pm = y_1 \pm y_2$. $\langle\delta J(1)\delta J(2)\rangle$ is strongly peaked for small x_- and y_-, and satisfies

$$\left[\frac{\partial}{\partial z} + \frac{x_-}{L}\frac{\partial}{\partial y_-} - \frac{\partial}{\partial x_-}D_-\frac{\partial}{\partial x_-}\right]\langle\delta J(1)\delta J(2)\rangle = 2D\left(\frac{\partial}{\partial x}\langle J\rangle_{\rm av}\right)^2, \quad (9)$$

where

$$D_- = 2\int (dk_y/2\pi)\int (dk_z/2\pi)[1-\cos(k_y y_-)]\langle b^2\rangle_{k_y,k_z} R(k_z - k_y x/L). \tag{10}$$

Here, D_- describes the correlated diffusion of two field lines. D_- vanishes for small x_-, y_- separation (field lines diffuse together), and becomes the sum of the uncorrelated diffusion coefficients when the separation becomes large (field lines diffuse independently). Eq. (9) has an approximate solution of the form

$$\langle\delta J(1)\delta J(2)\rangle = 2z_c(x_-,y_-)D(\partial\langle J\rangle_{\rm av}/\partial x)^2, \tag{11}$$

where $z_c(x_-,y_-)$ is the two-line correlation length.[1,2] If we approximate

$$D_- \sim k_0^2 y_-^2 D, \tag{12}$$

where k_0, the average k_y wave number, is

$$k_0^2 = (2D)^{-1}[\partial^2 D_-/\partial y_-^2]_{x_-=y_-=0}, \tag{13}$$

we can solve the moment equations of (9) for $z \gg z_0$ to give $\langle y_-(z)^2\rangle \cong (y_-^2 + 2x_-y_-z_0/L + 2x_-^2 z_0^2/L^2)\times\exp(z/z_0)$, so that

$$z_c(x_-,y_-) \cong z_0 \ln[3k_0^{-2}/(y_-^2 - 2y_-x_-z_0/L + 2x_-^2z_0^2/L^2)], \tag{14}$$

where $z_0 = (4k_0^2D/L^2)^{1/3}$. Setting $k_y = k_0$ in the definition of z_t, we find that $z_0 = (12)^{-1/3}z_t$.

The quantity $\langle\delta J(1)\delta J(2)\rangle$ is the total fluctuation correlation function. Since $\delta J = J^{(c)} + \tilde{J}$, where $J^{(c)}$ are uncorrelated at this stage of the calculation, the incoherent correlation function can be calculated by subtracting the coherent correlation function from $\langle\delta J(1)\delta J(2)\rangle$. The coherent correlation function is obtained by multiplying $J^{(c)}(1)$ by $J^{(c)}(2)$. It follows that, for $1\sim 2$, $\langle J^{(c)}(1)J^{(c)}(2)\rangle \sim z_t D(\partial\langle J\rangle_{\rm av}/\partial x)^2$. After subtracting this from (11), we obtain

$$\langle\tilde{J}(1)\tilde{J}(2)\rangle \sim 2z_0 D(\partial\langle J\rangle_{\rm av}/\partial x)^2 \ln[3k_0^{-2}c/(y_-^2 - 2y_-x_-z_0/L + 2x_-^2z_0^2/L^2)], \tag{15}$$

where $\ln c^2 = (12)^{1/3}$. Equation (15) states that the stochastic field lines in the presence of $\partial\langle J\rangle_{\rm av}/\partial x$ create current filaments of scale $L/k_0z_0 \sim \Delta x_t$ in x, k_0^{-1} in y, and z_0 in z. In (15) each fluctuation is evaluated at the same value of z (i.e., $z_1 = z_2$). However, the $z_- = z_1 - z_2$ dependence is easily incorporated by noting that the current fluctuation approximately follows the unperturbed field lines. Thus the z dependence can be included in (15) by replacing y_- with $y_- - (x_+/2L)z_-$. After this is done, (15) is Fourier transformed in y_- and z_- (denoted by $\langle\rangle_{k_y,k_z}$). Since (15) is a sharply peaked function in x_- of width Δx_t, we approximate the x_- dependence with a delta function. Finally we obtain

$$\langle\tilde{J}(1)\tilde{J}(2)\rangle_{k_y,k_z} = [2L(2\pi)^2/k_y^2][1 - J_0(1.4k_y/k_0)]D(\partial\langle J\rangle_{\rm av}/\partial x)^2\delta(x_-)\delta(k_z - k_y x/L), \tag{16}$$

where J_0 is a Bessel function. We can now use (7) and the Green's function to express the perturbed flux ($\delta\psi$) correlation function in terms of the current correlation function

$$\langle\delta\psi^2(x_s)^2\rangle_{k_y,k_z} = [(4\pi)^4L^2/2|k_y|]|G(x_s,x_s)_{k_y,k_z}|^2 D(x_s)(\partial\langle J\rangle_{\rm av}/\partial x)^2[1 - J_0(1.4k_y/k_0)]. \tag{17}$$

The fact that (17) contains G evaluated at $x = x' = x_s$ results from the δ functions in (16). Actually, these δ functions should be resonance functions of finite width Δx_t. We have neglected this, since in the "constant-ψ approximation",[9] G is approximately constant over this distance. Equation (17) contains D and k_0^2 which are determined by (6) and (13). Since $bB_0 = -\partial\delta\psi/\partial y$, we use (17) in (6), and (10) and (17) in (13) to produce two equations that determine D and k_0^2:

$$\left.\begin{matrix} D \\ k_0^2 \end{matrix}\right\} = 2\pi\left(\frac{4\pi L\,\partial\langle J\rangle_{\rm av}/\partial x}{B_0}\right)^2\int_{-\infty}^{\infty}\frac{dk_y}{|k_y|}\left\{\begin{matrix} 1 \\ k_y^2/k_0^2 \end{matrix}\right\}|G(x_s,x_s)_{k_y,k_{zr}}|^2\left[1 - J_0\left(\frac{1.4k_y}{k_0}\right)\right], \tag{18}$$

where $k_{zr} = k_y x_s/L$ and G is an implicit function of D. Unfortunately, the integral in (18) for k_0^2 diverges logarithmically, since for large k_y, $|G|^2 \sim |\Delta'|^{-2}$ and $\Delta' \sim k_y$. This divergence in the k_0^2 equation occurs because D_- increases more rapidly for small y_- than is assumed in (12). The proper procedure would be to solve for the y_- dependence of $D_-(y_-)$ as in Ref. 13, but this is too complicated for our purposes here. Instead we ignore the k_0^2 equation in (18) and use Eq. (8), the $D = 0$ value of the Green's function, in (18) to determine k_0. A numerical solution of (18) for k_0^2 then gives $k_0L \sim 6$.

The behavior of the average current $\langle J \rangle$ in the presence of the self-consistent, stochastic fields can be determined by ensemble averaging (1a) and neglecting the $B_y\, \partial J/\partial y$ nonlinearity. The result is $\partial\langle J\rangle_{\rm av}/\partial z = -\,\partial\langle bJ\rangle_{\rm av}/\partial x$. This can be evaluated from the correlation functions following the procedure of Ref. 6. A Lenard-Balescu–like equation for the evolution of $\langle J\rangle_{\rm av}$ results, and has the form $\partial\langle J\rangle_{\rm av}/\partial z = (\partial/\partial x)D\,\partial\langle J\rangle_{\rm av}/\partial x - (\partial/\partial x)F\langle J\rangle_{\rm av}$. Here, D [given by (6)] and F are magnetic field line diffusion and "drag" coefficients, respectively. Just as in the one-dimensional theory of plasma turbulence,[6] it can be shown that the diffusion and drag terms cancel exactly (at least for low turbulence levels), leaving $\partial\langle J\rangle_{\rm av}/\partial z = 0$. This important result ensures that the average current density and magnetic field will be z independent.

The time dependence of the fluctuation spectrum can be obtained by multiplying (1c) by ψ and ensemble averaging. This yields $\partial\langle\psi(1)\psi(2)\rangle/\partial t = \eta[\langle\psi(1)J^{(c)}(2)\rangle + \langle\psi(1)\tilde{J}(2)\rangle]$. It is straightforward to show that the resulting time dependence is exponential but with a small growth rate of order of the inverse macroscopic resistive time $L^2\eta^{-1}$.

In conclusion we believe that the methods and concepts presented here may be of use in a variety of stochastic magnetic field problems.

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