attractor in a low-dimensional dynamical system.[14] The situation at large aspect ratio is not yet clear.

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Holographic Imaging without the Wavelength Resolution Limit

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It is usually assumed in both optical and acoustical holography that the resolution of a reconstructed image is limited by the wavelength of the radiation. In this paper it is demonstrated that this is not necessarily true in acoustical holography. A technique which images the source vector intensity as well as the sound pressure amplitude with a resolution independent of the wavelength is presented. Application to electromagnetic radiation may be possible.

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It is usually assumed in holography that the spatial resolution in a reconstructed image is limited by the wavelength of the radiation. This is certainly true in optical holography and is usually assumed in acoustical holography, with the result that acoustical holography has not been used in otherwise natural applications such as low-frequency–noise abatement and musical-instrument research. However, the wavelength resolution limitation is not intrinsic to the fundamental theories of holography but rather is due to experimental limitations which are present in optical holography but are not necessarily present in acoustical holography. In this paper we demonstrate experimentally that it is possible to obtain high-resolution images of sound sources regardless of the wavelength, and that a single measurement is sufficient to reconstruct the source sound pressure, the particle velocity, the far-field radiation pattern, and most importantly the source vector intensity. Our results should form a basis for the development of powerful new tools for research in both acoustic and electromagnetic radiation, although the latter would require more consideration of the vector nature of the field.

The theory underlying non–wavelength-limited acoustic imaging is as follows: One assumes a monochromatic source region (radiating into a linear and homogeneous medium with a wavelength λ) located to one side of an infinite plane I defined by $z=z_I$. From Green's theorem, the sound pressure amplitude and phase in the plane I [represented by the complex quantity $P_I(x',y')$] can be used to calculate[1] exactly the sound field P at any field point (x,y,z) to the side of the plane away from the sources $(z>z_I)$:

$$P(x,y,z) = \iint P_I(x',y')G(x-x',y-y',z-z_I)dx'dy', \quad (1)$$

where

$$G(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial\alpha}\frac{\exp[i(2\pi/\lambda)(x^2+y^2+\alpha^2)^{1/2}]}{(x^2+y^2+\alpha^2)^{1/2}}\bigg|_{\alpha=z}. \tag{2}$$

Physically Eq. (1) represents forward propagation of sound away from the sources.

If one now restricts the field point to lie in a second plane H defined by $z=z_H=z_I+d$, then Eq. (1) becomes an explicit two-dimensional convolution integral which can be inverted with use of the convolution theorem. Defining $P_H(x,y)=P(x,y,z_H)$ and $G_d(x,y)=G(x,y,d)$, and letting $\hat{f}(k_x,k_y)$ denote the two-dimensional Fourier transform of $f(x,y)$, we have from Eq. (1) $\hat{P}_H(k_x,k_y)=\hat{P}_I(k_x,k_y)\hat{G}_d(k_x,k_y)$. The transform $\hat{G}_d$ is found analytically

$$\hat{G}_d(k_x,k_y) = \begin{cases}\exp\{id[(2\pi/\lambda)^2-k^2]^{1/2}\}, & k\le 2\pi/\lambda,\\ \exp\{-d[k^2-(2\pi/\lambda)^2]^{1/2}\}, & k>2\pi/\lambda,\end{cases} \tag{3}$$

where $k^2=k_x^2+k_y^2$. It should be noted that $\hat{G}_d$ for $k<2\pi/\lambda$ represents the radiation of sound into the far field, and $\hat{G}_d$ for $k>2\pi/\lambda$ represents the rapid exponential decay of the nonradiating near field of the sources, composed of evanescent waves.

Dividing the expression for $\hat{P}_H$ by $\hat{G}_d$ and taking the inverse Fourier transform (indicated by F^{-1}) we have

$$P_I(x,y)=F^{-1}\{\hat{P}_H(k_x,k_y)[\hat{G}_d(k_x,k_y)]^{-1}\}. \tag{4}$$

Thus knowledge of the sound pressure in the H (hologram) plane can be used to deduce exactly the pressure in the I (image) plane. If the sound sources are reasonably coplanar, then the plane I is made to coincide with the source plane for the reconstruction. The technique may be extended for nonplanar and nonmonochromatic sources.

Equation (4) differs from an expression representing optical holography in that $\hat{G}_d^{-1}$ contains an exponentially increasing part for $k>2\pi/\lambda$ which takes the strongly decayed (but nonzero) evanescent-wave components and restores them to their exact values in the image plane. Physical reconstruction methods (e.g., those with coherent light) must use forward-propagating waves, which would be represented by $\hat{G}_d{}^*$, the complex conjugate of $\hat{G}_d$. This "propagator" retains the exponentially decaying part, and for $d>\lambda$, the decay is so rapid that essentially all of the information for $k>2\pi/\lambda$ is lost, and without these high spatial frequencies the image resolution is limited. In order to have no such resolution limit, Eqs. (4) must be evaluated numerically. However, it is then necessary that P_H faithfully represent the evanescent-wave information. In optical holography the wavelengths are so short that the hologram is necessarily recorded many wavelengths from the sources with the result that the evanscent-wave amplitudes fall well below the noise level of the holograms recording medium.[1] Thus, if Eq. (4) were used, the exponentially increasing part of $\hat{G}_d^{-1}$ would yield meaningless results. In order to use Eq. (4), the hologram must be recorded sufficiently close to the sources so that the evanescent-wave information falls within the dynamic range of the detector. It can be shown that the resolution R (distance between two resolvable point sources) of the reconstruction is given by[2]

$$R=(20\pi/D\ln 10)d, \tag{5}$$

where D is the dynamic range of the recording medium in decibels. Equation (5) does not contain λ and indicates that good resolution is obtained when d is small, that is, when the hologram is recorded as close as possible to the sources. For low-frequency sound (or electromagnetic) radiation it is experimentally possible to have d sufficiently small so that $R \ll \lambda$. It should be noted that having a small d virtually eliminates the effects of a finite hologram aperture.[2] Another important aspect is that the dynamic range D, which must include all effects which result in a loss of precision in the measurement of P_H, also limits the resolution of the reconstructions. Thus applications of this technique may require more precision in amplitude measurement than in the case of conventional holography.

When d is small, both P_H and P_I are in the near field and it may seem that little would be gained in the reconstruction. However, our experimental results show that a significant increase in clarity can be obtained even when $d \approx 0.04\,\lambda$. More importantly, much more information (in fact everything) can be calculated from the near-field measurement of P_H. Replacing $\hat{G}_d^{-1}$ with $\hat{G}_{d'}$ in Eq. (4) yields the sound pressure at a distance $d+d'$ away from the sources; large d' yields the far-field pressure. The expression for the particle velocity $\vec{u}$ in both the near field and far field can be obtained by taking the gradient of the right-hand side of Eq. (4) and noting that $\partial/\partial z = -\partial/\partial d$ for the near field, and $\partial/\partial z = +\partial/\partial d'$ for the far field. The near field $\vec{u}$ can be used to obtain the modal structure of a vibrating surface. The vector intensity $\vec{S} = \frac{1}{2}\operatorname{Re}(P\vec{u}^*)$ can also be calculated. The directivity pattern of the sources is found from $\vec{S}$ in the far field. Of primary importance is the determination of $\vec{S}$ in the near field, because this is what images the true energy-producing sources; the near-field sound pressure (or its square) does not yield this information. Usually only radiation perpendicular to the source plane is of interest so that calculating S_z is sufficient. The total power radiated and the source-radiation resistance and reactance can also be determined.

Because of the exponentially increasing part of $\hat{G}_d^{-1}$ and the numerical derivative required for $\vec{u}$ (and $\vec{S}$), there are potential problems with noise, finite-aperture effects, numerical roundoff, etc., and it is important to verify the practicality of Eq. (4) and related expressions. We have accomplished this with an acoustic holographic apparatus consisting of a planar microphone array of 256 elements (inexpensive moving-coil devices) spaced 20.3 cm apart in a 16×16 grid, resulting in a 325-cm square hologram aperture. The array and the sound sources to be measured are housed in a quasianechoic chamber made out of 5-cm-thick fiberglass tiles. The microphone signals are routed through a 256:1 multiplexer to a lock-in amplifier which detects the in-phase and quadrature parts of the signal. The necessary reference signal is taken from a separate microphone near the sources. The microphone array scans the sources with a 4×4 pattern of 5-cm increments to produce a 64×64 data array which represents P_H discretely sampled every 5 cm. The data are fed directly into an on-line minicomputer which evaluates Eq. (4) and related expressions with a fast-Fourier-transform algorithm. The various reconstructed images (pressure, intensity, etc.) are displayed with computer graphics on either a color TV monitor or a four-color plotter. Further details of the apparatus will be published elsewhere.[3]

To test the technique we used as sources two small unbaffled speakers (5 cm diameter) mounted 20 cm apart. The two speakers were driven in phase at 220 Hz, corresponding to a 156 cm wavelength in air; it was assumed that the speakers would act as point sources. The speakers were situated in a plane 10 cm away from the microphone array. Figure 1(a) shows the sound pressure amplitude $|P_H|$ measured by the array. Even with the array as close as 0.06λ the sound field is significantly spread out. It should be noted that most of the irregular structure in the data is due to the lack of calibration of the inexpensive microphones. This unfortunately gives our recording system an effective dynamic range of only 30 dB. Figure 1(b) shows the result of trying to image the sources with the customary holographic technique, e.g., by assuming that no evanescent-wave information is recorded. The resolution is clearly limited by the sound wavelength. With the customary technique, the reconstructed velocity at the source plane is nearly proportional to P and hence yields no significant information about the sources. Figure 1(c) shows the reconstruction of the intensity S_z with use of Eq. (4) and related expressions. The two point sources are clearly resolved and could probably be resolved even if only 10 cm apart, as predicted

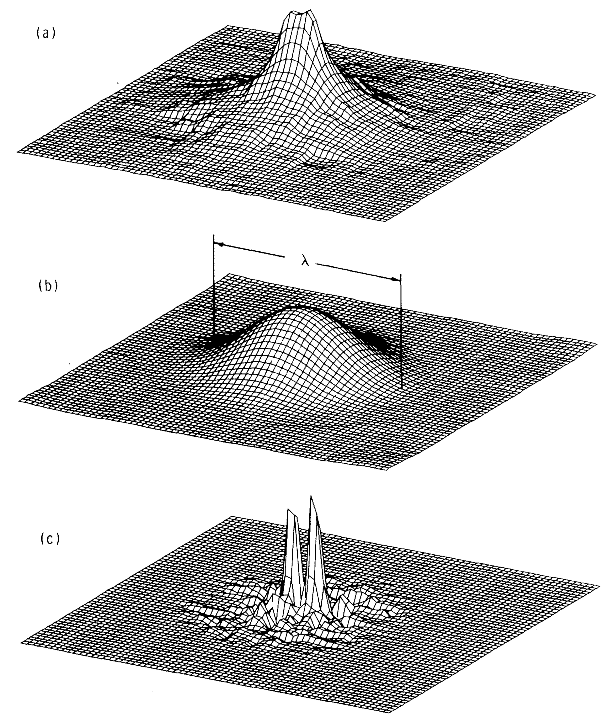

FIG. 1. Holographic data for two point sources. (a) Sound-pressure amplitude in the hologram plane. (b) Conventional holographic reconstruction of the sources. The resolution-limiting wavelength λ is indicated. (c) Reconstruction of the source intensity with use of our technique. The two point sources are clearly resolved.

by Eq. (5). Reconstruction of the pressure and velocity are similar to Fig. 1(c). The actual amplitude of the peaks depends on the details of the data processing; this will be discussed in a future paper.[2]

We also tested the technique with a noise source consisting of a blower mounted onto the side of a wooden box (31 cm×41 cm×29 cm) which was open on one side. A harmonic of the blower's rotational frequency, 120 Hz, excited a resonance in one side of the box. A spearate microphone monitored this frequency with a 50 Hz bandwidth and provided the reference for the hologram. The open side of the box faced the microphone array and was a distance of 10 cm from the plane of the array. Figure 2(a) shows the reconstructed sound pressure in the plane of the box; the inset, representing a top view of the three-dimensional figure, shows an outline of the box and the relative amplitude and phase of the sound pressure. A dipolar pressure field generated by the resonating side of the box is clearly apparent. The large sound-pressure amplitude near this side is observed by a near-field listener. However, we emphasize that this does not represent the energy-producing sources. Figure 2(b) shows the reconstructed intensity S_z in the plane of the box. From the inset it can be seen that the source intensity pattern is quite different from that of the pressure. Inside the box there is a source ($S_z > 0$) and a sink ($S_z < 0$, corresponding to a region where the sound pressure and particle velocity have a phase difference greater than 90°). This asymmetry inside the box is caused by the presence of a block of Styrofoam wedged across the width of the box which was used to dampen the resonance; the positive intensity peak is directly over the Styrofoam block. There is no evidence of the Styrofoam block in the pressure-amplitude reconstruction. Apparently surfaces which control the relative phase of the sound pressure and particle velocity determine the energy-producing regions.

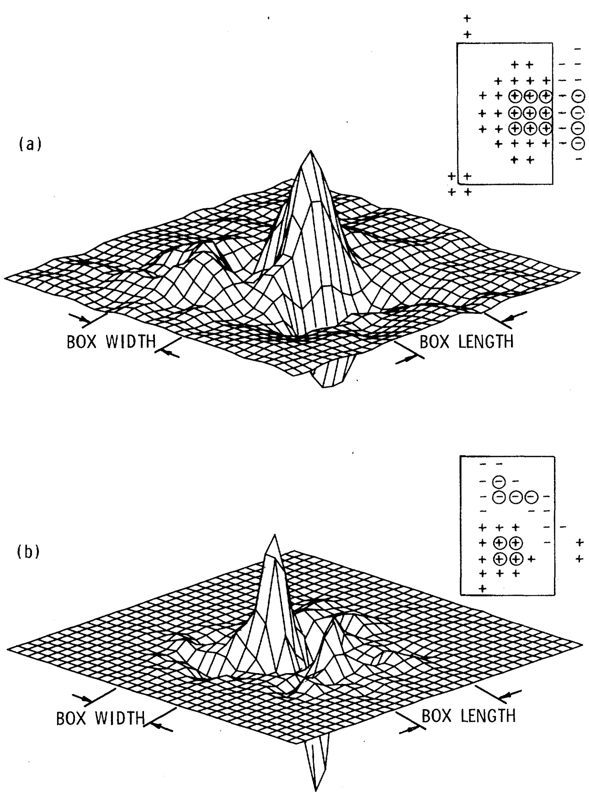

FIG. 2. Holographic reconstructions of a noise source consisting of a wooden box excited by a blower. Note that the three-dimensional figures represent an area which exceeds the area of the box. The insets show an outline of the box and some relative amplitudes and phases. (a) Reconstructed source sound pressure, showing dipole nature of the sound field. (b) Reconstructed source intensity, showing evidence of a Styrofoam block inside the box which is not apparent in the pressure-amplitude reconstruction.

In addition to illustrating the significant information content of the intensity reconstruction, Fig. 2 demonstrates the increase in resolution over conventional holography; the resolution of this figure is ~10 cm, while the wavelength at 120 Hz is 286 cm, nearly the size of the hologram aperture.

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