Crossover from Weak to Strong Coupling in SU(N) Lattice Gauge Theories

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Coupling constant renormalization for SU(N) lattice gauge theories is studied with Hamiltonian methods. Expansions to $O(g^{-16})$ suggest that the crossover from weak to strong coupling occurs progressively more abruptly in the variable $\lambda^2 = g^2N$ as N increases. The constants relating the string tension to the scale-breaking parameter Λ are estimated for all N. It is suggested that the Callan-Symanzik β function develops a kink in the $N \to \infty$ limit at a nonzero value of λ^2.

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SU(N) gauge theories are of considerable interest for several reasons. The $N = 3$ theory is believed to describe the pure gauge-field sector of hadron physics.[1] Gauge theories with $N > 3$ may be relevant to grand unification schemes.[2] And finally, there is hope that the $N \to \infty$ limit of these theories is exactly soluble.[3]

Considerable progress has been achieved in understanding the dynamics of SU(2) and SU(3) gauge theories through their lattice formulations. This approach makes it possible to do systematic nonperturbative calculations. Computer simulations of SU(2),[4] and strong-coupling expansions of SU(3),[5] suggest that these theories, which are asymptotically free, confine static quarks. These calculations have shown, roughly speaking, that gauge theories are weakly coupled at short distances but cross over to strong-coupling behavior quite abruptly as larger distances are considered. We wish to discuss this crossover behavior for all SU(N) gauge groups and point out some interesting systematic trends as N is varied. Our calculations indicate that the crossover region in the coupling $\lambda^2 = g^2N$ becomes progressively narrower as N increases. The constants C_N relating each theory's string tension T to its weak-coupling scale-breaking parameter Λ are estimated for all N. We find that C_N increases significantly with N. We compute Callan-Symanzik β functions for all N and all coupling. The resulting curves suggest that a kink develops in the crossover region as $N \to \infty$ and that it occurs at a limiting, nonzero, value of $\lambda^2 = g^2N$ which we estimate. If real, this nonanalyticity would imply that a $1/N$ expansion for SU(N) lattice gauge theories contains pathologies.

The calculations leading to these results are a modest extension of SU(3) calculations which have been discussed in part elsewhere.[5] Consider the Hamiltonian of SU(N) gauge fields in 3 + 1 dimensions,[6]

$$H = (g^2/2a)\{\textstyle\sum_l \vec{E}_l^{\,2} - x \sum_p \mathrm{tr}[U(p) + \mathrm{H.c.}]\}, \quad x = 2/g^4, \tag{1}$$

where $\vec{E}_l^{\,2}$ is the quadratic Casimir operator of SU(N) on the link l of a three-dimensional cubic lattice, and $U(p)$ is the unitary fundamental representation of the product of group elements on the boundary of a plaquette p. Quantities of physical interest can be calculated by use of Eq. (1). For example, the coefficient of the linear term in the potential between two widely separated quarks [sources and sinks of flux in the fundamental representation of SU(N)] can be computed in a power series in $x = 2/g^4$. This quantity, the "string tension," is the order parameter of the theory and has been used to renormalize the SU(3) theory.[5] The requirement that the string tension be independent of the lattice spacing a determines the dependence of the coupling constant g on a. In particular, for SU(N) Eq. (1) gives an expansion for the string tension,

$$T = \frac{g^2N}{2a^2}\frac{N^2-1}{2N^2}W(x), \tag{2a}$$

where

$$W(x) = \sum_{n=0}^{\infty} t_n x^n, \quad t_0 = 1, \tag{2b}$$

and the coefficients t_i can be found with ordinary perturbation theory. The theory's β function is then[5]

$$\frac{\beta(g)}{g} = -\frac{d\ln g}{d\ln a} = \frac{-1}{1-2xW'/W}. \tag{3}$$

The series in Eq. (2b) has a finite radius of convergence and can be treated by standard extrapolation methods to probe the weak-coupling region of the theory. Recall that for SU(3) the β function computed in this way matched the continuum theory's weak-coupling β function at $g \sim 0.9$.[5]

The calculation of Eqs. (2) and (3) has been carried out for all SU(N) theories to $O(g^{-16})$. Although these are relatively low-order calculations—the expansion for SU(3) now exists to $O(g^{-24})$—we will *assume* that they are sufficient to describe each theory's crossover region. This assumption is plausible because of the success of the SU(3) calculation—higher orders show that the match of the weak- to strong-coupling expansions near $g \approx 0.9$ is smooth but the higher orders do not change the low-order estimates of the shape of the β function above this matching region. In effect, we will be assuming that the SU(N) theories for $N>3$ confine static quarks in their continuum limits. Since the elaborate calculations for SU(2) and SU(3) support this hypothesis, it is reasonable to apply it to groups of greater dimensionality which have more intrinsic disorder. Calculations beyond $O(g^{-16})$ could be done to establish this assumption.

Our strong-coupling expansions will be renormalized by holding the string tension T fixed as the lattice spacing is varied. This is an unconventional, nonperturbative renormalization condition. A more conventional method considers scale-breaking effects at weak coupling and uses a quantity Λ with dimensions of a mass to set the scale of deviations from free-field behavior. Since an SU(N) gauge theory can have but one scale to characterize its continuum limit, $\sqrt{T}$ and Λ must be proportional,

$$\sqrt{T} = C_N \Lambda. \tag{4}$$

For SU(3), Λ is measurable in deep inelastic scattering, and $\sqrt{T}$ is known from heavy quark spectroscopy, so that a nonperturbative calculation of C_3 is of considerable interest. But the real world contains additional degrees of freedom—light quarks—which are not incorportate in the present lattice calculation, and so only a semiquantitative comparison with experiment is warranted. Anyway, the dependence of C_N on N is also of interest in understanding gauge-field dynamics.

To determine C_N we need the weak-coupling dependence of the theory's mass scales on the lattice coupling constant. This follows from the weak-coupling β function,[7]

$$\frac{\partial \ln g}{\partial \ln a\Lambda_{\rm SL}} = -\frac{\beta(g)}{g} = \beta_0 g^2 + \beta_1 g^4 + \dots, \tag{5a}$$

where

$$\beta_0 = \frac{11}{3}\frac{N}{16\pi^2}, \quad \beta_1 = \frac{34}{3}\frac{N^2}{(16\pi^2)^2}. \tag{5b}$$

The subscripts "SL" on Λ remind us that we are using a "spatial lattice" regulator. It follows from Eq. (5a) that

$$\Lambda_{\rm SL} = \frac{1}{a}\left(\frac{48\pi^2}{11\lambda^2}\right)^{51/121} \exp\left(\frac{-24\pi^2}{11\lambda^2}\right)[1 + O(\lambda^2)], \tag{6}$$

where we see that the natural expansion parameter for the weak coupling features of each SU(N) theory is $\lambda^2 = g^2 N$. Since Eq. (5a) does not determine the scale of $a\Lambda_{\rm SL}$ we have set a convention in Eq. (6) which is borrowed from deep-inelastic phenomenologists.[8] With the choice of constants in Eq. (6), the lattice coupling varies with the lattice spacing as

$$g^2 = \frac{1}{\beta_0 \ln(a\Lambda_{\rm SL})^{-2} + \beta_1/\beta_0 \ln\ln(a\Lambda_{\rm SL})^{-2}} + \dots. \tag{7}$$

This formula shows the virtue of Eq. (6)—the two-loop correction to the β function does not induce a rescaling of $\Lambda_{\rm SL}$ as a varies.

We have taken the strong-coupling expansions for $a^2 T$ and matched onto the weak-coupling scaling law Eq. (6) for each N. For each N both the Taylor series and its various Padé approximants were used and compared. These different procedures resulted in the theoretical uncertainties noted in numerical estimates below. No matter how the strong-coupling calculations were used, the same trends in the N dependence were obtained. In doing these analyses it was important to discover that $z = 1/g^2 N$ is the natural expansion parameter for the strong-coupling series Eq. (2b). By explicit calculation of the series one finds that if they are written in the form

$$W(z) = \sum_{l=0} \omega_l z^l, \quad \omega_0 = 1, \tag{8}$$

then each ω_l has a finite limit as $N \to \infty$.[9] This means that, at least through the order we have calculated, the strong-coupling β function has a nontrivial $N \to \infty$ limit when expressed as a series in z. The values for ω_l are listed in Table I. These coefficients were calculated *exactly* for each N. This required developing the Clebsch-

TABLE I. String-tension expansions and the constants C_N for SU(N) gauge theories. $\omega_0 = 1$ for all N and other unlisted coefficients of order $i < 8$ vanish identically.

N	ω_4	ω_8	C_N
2	−65.01587	−5140.05734	80± 17
3[a]	−17.47059	−1000.80194	240± 40
5	−13.62724	−472.33131	561± 172
6	−13.09684	−426.98544	607± 174
30[b]	−12.04122	−347.69984	773± 239

[a] ω_6 is nonzero for SU(3) and can be found in Ref. 5.
[b] These coefficients are within a fraction of a percent of the $N \to \infty$ limiting values.

Gordan series for SU(N) sufficient to decompose the product of six U matrices on a link. The N 2, 3, and 4 calculations required special care.[9] The N =3 results were checked against other SU(3) calculations[5] which used different techniques. The details of these analyses and the exact coefficients will be discussed elsewhere.

The matching of the strong- and weak-coupling string-tension calculations is shown in Fig. 1. We observe that these calculations are consistent with our assumption that there is a smooth transition from weak to strong coupling for each N. From these curves we read off the values of C_N listed in the table. The value which may be of most immediate interest is, for SU(3),

$$\sqrt{T} = (240 \pm 40)\Lambda_{SL}. \tag{9}$$

Higher-order SU(3) calculations give additional weight to this result.[5] The uncertainty in Eq. (9) reflects the fact that the C_N are exceedingly sensitive to the coupling of the crossover region. Two comments concerning these results are in order. First, the C_N increase with N. This trend is a consequence of the SU(N) *matrix* character of the link variables. Recall that for O(N) nonlinear sigma models in 1 +1 dimensions, the analogs of C_N *decrease* with N and approach a finite limit as $N \to \infty$.[10] And second, the huge size of C_N is partially an artifact of use of a spatial lattice to do perturbation theory. Typically momentum-space renormalization methods are used in deep inelastic phenomenology. With each method there is a different Λ. The relation between momentum space (Λ_{mom}) regularization and a Euclidean lattice (Λ_{EL}) has been computed[11]:

$$\Lambda_{mom}/\Lambda_{EL} = 83.5 \tag{10}$$

for SU(3). The relation between Λ_{EL} and Λ_{SL} must be computed,[12] however, before a useful relation between $\sqrt{T}$ and Λ_{mom} is obtained from this calculation. Presumably the large change in scale in Eq. (10) absorbs much of the change in scale in Eq. (9).

The important trends in Fig. 1 are made more visible if the β functions are plotted for each N. With use of Eq. (3) the β functions of Fig. 2 were obtained numerically from Fig. 1. Note that as N increases the crossover regions from weak to strong coupling become narrower and move to a

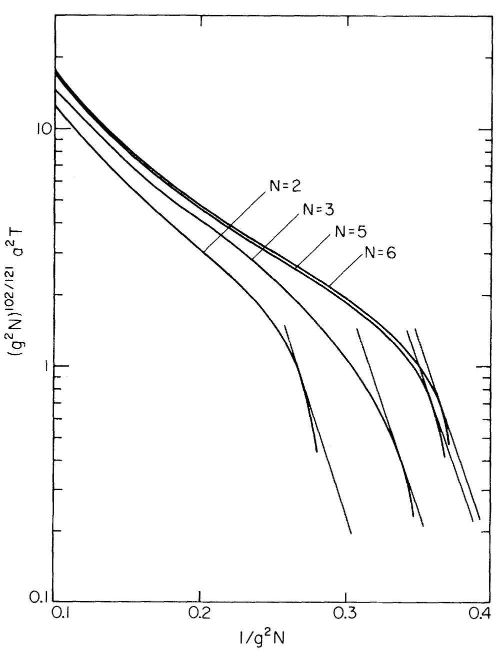

FIG. 1. Matching strong and weak string-tension expansions for N = 2, 3, 5, and 6. The parallel lines are from Eq. (6) and the curves are the $O(g^{-16})$ strong-coupling expansions.

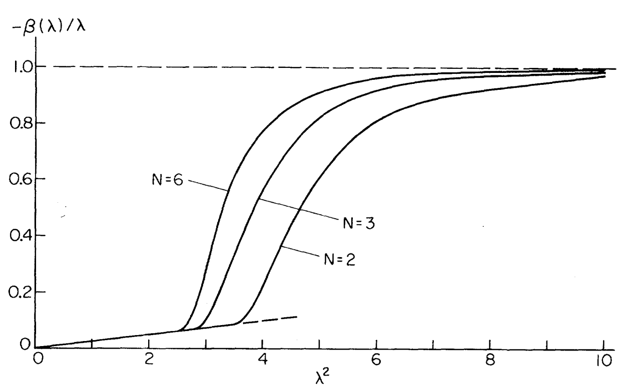

FIG. 2. β functions for N = 2, 3, and 6.

small but nonzero coupling. The curves suggest that as $N \to \infty$ the β function develops a kink at $\lambda^2 = 2.7$.[13] Transitions of this sort have been obtained in studies of one-plaquette models of lattice theories.[14]

We conclude this article with an observation concerning the reliability of finite lattice calculations of the crossover regions. Note from Fig. 1 that the value of a^2T where the strong-coupling calculation matches the weak-coupling scaling law is 0.3–0.4, and is roughly independent of N. But $(a^2T)^{-1/2}$ is a measure of each system's linear correlation length, and it is about 1.6–2.0 when measured in units of the lattice spacing. This is quite small so that ordinary methods of analysis such as strong coupling expansions, finite size scaling, computer simulations, and analytic solutions of small systems should be adequate here.

The calculations presented in this article are just exploratroy. As explained in the last paragraph, they appear to be at least reasonable and self-consistent. It would be good to calculate higher orders and to run computer simulations for various SU(N) gauge groups.

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Flavor Goniometry by Proton Decay

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Unification of strong and electroweak forces implies that protons (and bound neutrons) decay. Modes like $\bar{\nu}\pi$, $e^+\pi$, and μ^+K^0 are expected, while modes like $\mu^+\pi$ and e^+K^0 are "Cabibbo" suppressed. Branching ratios can reveal much about the nature of the unifying group and the origin of fermion masses. Plausible models of unification and flavor mixing give surprisingly different predictions for two-body branching ratios.

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Once, weak decays of hadrons involved only one flavor-mixing parameter—the Cabibbo angle.[1] With N-fermion families, $(N-1)^2$ parameters describe the flavor mixing of electroweak phenomena.[2] Most electroweak-chromodynamic unifications predict a new interaction which violates the conservation of baryon number. Protons and bound neutrons decay into states with $S=0$ or 1 containing one antilepton (e^+, μ^+, or $\bar{\nu}$). Branching ratios of these $\Delta B = \Delta L = -1$ decays are *not* necessarily determined by the $(N-1)^2$ flavor-mixing parameters, a point stressed by many

