Energy Principle with Global Invariants for Toroidal Plasmas

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A variational principle is proposed for constructing equilibria with low free energy in toroidal plasmas in which relaxation is dominated by a tearing mode of single helicity. States with current density vanishing on the boundary are constructed. Theoretical predictions are compared with experimental data from reversed field pinches and tokamaks.

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The formulation of a variational principle for a complete class of static equilibria of toroidal plasmas is due to Kruskal and Kulsrud.[1] They characterized equilibria for ideal plasmas by a nondenumerable set of topological invariants derivable from the ideal hydromagnetic equations of motion. A laboratory plasma, however, is inevitably subject to nonideal effects such as those associated with resistivity or microturbulence. Taylor[2] has conjectured that the global invariant $K=\int_{V_0} d^3r\, \vec{A}\cdot\vec{B}/2$, first introduced in the astrophysical literature by Woltjer[3] for a perfectly conducting plasma, remains an invariant even in the presence of a small but finite amount of dissipation. By minimizing the energy $W=\int_{V_0} d^3r \times B^2/2$ subject to the invariant K, Taylor has argued that a toroidal discharge, initially violently unstable, may relax into a force-free equilibrium state given by $\vec{j}=\lambda\vec{B}$, where λ is a constant. Taylor has provided no detailed justification for K conservation, but his theory has attracted much attention because it agrees satisfactorily with experimental observations[4] on field reversal from Zeta. Unfortunately, for tokamak discharges, where the toroidal field is approximately constant across the plasma, Taylor's theory predicts flat current profiles, which are usually not observed experimentally. Even in reversed-field pinches, the toroidal current is observed to be small near the wall[4] which in general violates the relation $\vec{j}=\lambda\vec{B}$. We interpret these observations to imply that the replacement of Kruskal and Kulsrud's infinity of constraints by a single one was too drastic a step; that a reasonably well-confined plasma preserves at least a few more approximate invariants over the time scale on which the growth and nonlinear development of tearing instabilities takes place. This time scale is, of course, short compared with the time scale of plasma transport, which is what determines the gross features of the current and pressure profiles. Thus we seek a variational principle which selects a special subset of the complete class of equilibria of Kruskal and Kulsrud, including those which can be sustained even on the transport time scale.

The shorter the time scale considered, the better preserved are many approximate invariants of motion. We are thus led naturally to consider the growth and decay of the fastest-growing tearing mode to be the mechanism responsible for the breaking of the ideal constraints. From linear and nonlinear theories of tokamak stability,[5] we know this mode to be the $m=1$, $n=1$ tearing mode. Indeed, there is experimental evidence[6] that discharges in the "internal sawtooth" regime in which the plasma exhibits soft $m=1$, $n=1$ activity uncoupled to weak higher harmonics (as opposed to conditions under which strong coupling to $m\geq 2$, $n=1$ modes leads to a major disruption with global flattening of the current profile, in accordance with Taylor's theory) are particularly favorable for confinement in tokamaks. Even during past experiments in pinches,[4] $m=1$ helices are observed prior to field reversal. For pinches the n number of the dominant mode should be such that the resonant surface falls within the plasma. However, predictions for F-θ trajectory and other qualitative features in this theory are not very sensitive to the choice of the dominant mode. We shall therefore, confine ourselves to the $m=1$, $n=1$ mode, given its importance for tokamaks.

In the following, we first assume the existence of a tearing mode of single helicity which grows from an axisymmetric state, saturates, and decays back to a new axisymmetric state. Although we are mainly considering the $m=1$, $n=1$ mode, it is instructive to allow a mode of arbitrary helicity. Within the quasi-ideal model,[7] we find that there is an infinite set of constants of the motion for each assumed helicity. The special role of the invariant K is confirmed by the observation that it is the sole occupant of the intersection of these sets. The model, which is described in Fig. 1, allows for compressible and incompressible displacements of the plasma. The contours

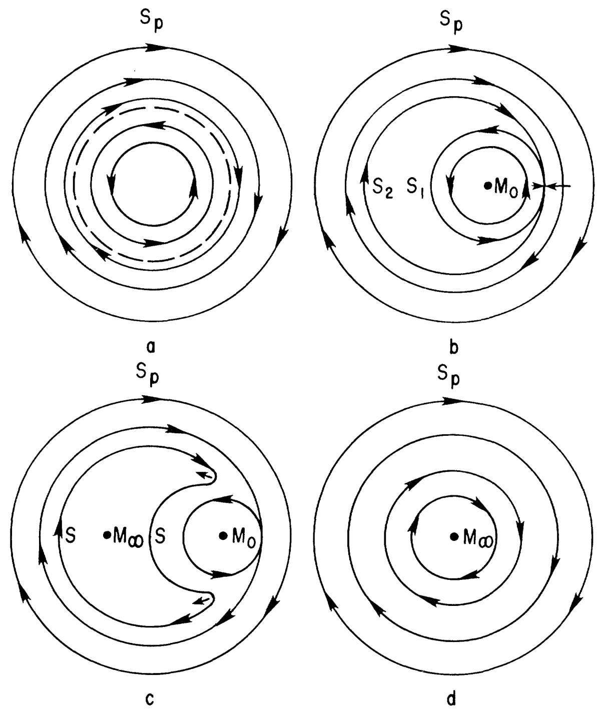

FIG. 1. Model of magnetic reconnection.

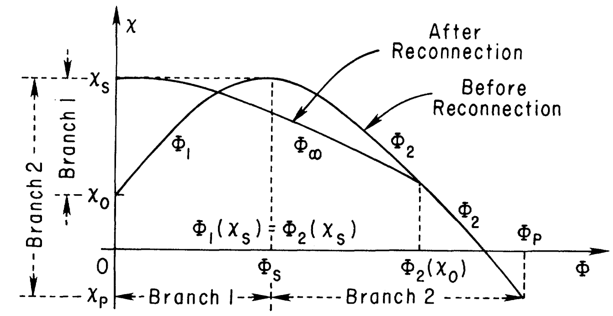

FIG. 2. Helical flux function $\chi(\Phi)$ before and after reconnection.

indicate the so-called auxiliary magnetic field $(B_\theta - rB_z/Rq_s)\hat{\theta}$ for a cylinder with periodicity length $2\pi R$ in the z direction which vanishes initially at the singular surface $q \equiv rB_z/RB_\theta = q_s$, shown by the dashed line in Fig. 1(a). In the initial state the plasma is assumed to be unstable to a helical perturbation of pitch q_s resonant at the singular surface. The argument is based purely on the assumed helical topology, and is thus valid also for a torus to the extent that the assumption of single helicity is valid. The plasma flows from the vicinity of the original magnetic axis, M_0, into a magnetic island with a new magnetic axis, M_∞. Reconnection occurs at the x point; otherwise the plasma is assumed ideal. Surfaces S_1 and S_2 [Fig. 1(b)], for example, merge to form surface S [Fig. 1(c)], conserving helical and toroidal (but not poloidal) flux. We consider closed helical field lines of the same pitch (q_s) as the separatrix, drawn on S_1, S_2, S, and S_p, the surface of the plasma in contact with the perfectly conducting wall. We define $2\pi m\chi_1$, $2\pi m\chi_2$, and $2\pi m\chi_\infty$ plus $2\pi\Phi_p$ as the fluxes crossing helical strips with one edge on S_p and the other edges on S_1, S_2, and S, respectively, where $2\pi\Phi_p$ is the total toroidal flux. χ_1, χ_2, and χ_∞ are surface quantities,[1] and are conserved on the time scale of the instability. Since S_1, S_2, and S share the separatrix at the instant of reconnection, we have $\chi_1 = \chi_2 = \chi_\infty$. During reconnection, the toroidal flux trapped between S_1 and S_2 remains trapped in S. Assuming that the toroidal flux function $\Phi = 0$ at M_0, we have $2\pi\Phi_\infty = 2\pi(\Phi_2 - \Phi_1)$, where $2\pi\Phi_1$, $2\pi\Phi_2$, and $2\pi\Phi_\infty$ are the toroidal fluxes enclosed by the surfaces S_1, S_2, and S, respectively. The total toroidal flux $2\pi\Phi_p$, enclosed by the plasma surface S_p, is a global invariant by virtue of the boundary conditions. The remaining surface quantity of interest is the poloidal flux function Ψ. We assume that $\Psi = 0$ on S_p. It is easy to see that the three surface quantities χ, Ψ, and Φ are linked by the relationship $\chi = q_s\Psi - \Phi$. The helical flux $\chi(\Phi)$ is shown in Fig. 2. Since $\Psi'(\Phi) = 1/q$ we have $\chi'(\Phi) = q_s/q - 1$. Initially then, χ has a maximum χ_s at $q = q_s$. In the final state [Fig. 1(d)], which has lower energy than the initial state,[7] χ is a monotonic function of Φ (Fig. 2). For the initial state, we obtain the double-valued function $\Phi(\chi)$ with branches Φ_1: $[\chi_0, \chi_s] \to [0, \Phi_s]$ and Φ_2: $[\chi_p, \chi_s] \to [\Phi_s, \Phi_p]$. The final toroidal flux function $\Phi_\infty(\chi)$ after reconnection is $\Phi_\infty(\chi) = \Phi_2(\chi) - \Phi_1(\chi)$ for $\chi \in [\chi_0, \chi_s]$ and $\Phi_\infty(\chi) = \Phi_2(\chi)$ for $\chi \in [\chi_p, \chi_0]$.

Following Greene and Johnson,[8] we represent the magnetic field $\vec{B} = \nabla\zeta \times \nabla\Phi(V) \times \nabla\theta(V) \times \nabla\theta$ in the coordinate system (V, θ, ζ). With the assumption that the scalar and vector potentials are single values, $\oint_C \vec{A} \cdot d\vec{l}$ must be constant in the time, whether the contour C is drawn on S_p in the toroidal or poloidal direction. These conditions are satisfied by the choice $\vec{A} = \Phi(V)\nabla\theta - \Psi(V)\nabla\zeta$, with Φ vanishing on the magnetic axis, and Ψ vanishing on S_p. [Since $\Psi(V_p) - \Psi(0)$ is not conserved during reconnection, K will not be conserved if we take $\Psi(0) = 0$, as concluded also by Kadomtsev.][7]

We consider now the functional

$$G[w]=\int_{V_0} d^3r\, w(\chi)\, \frac{\vec{A}\cdot\vec{B}}{2} = \frac{(2\pi)^2}{q_s}\int_{V_0} d\mu(\chi)[\Phi(\chi) - \tfrac{1}{2}\{\Phi(\chi)\chi\}'], \tag{1}$$

where $w(\chi)$ is an arbitrary function and $d\mu(\chi) = w(\chi)d\chi$ may be looked upon as an infinitesimal invariant measure convected by the plasma. Since $\Phi(\chi)$ is a double-valued function in the initial state, and single valued in the final state, we have to be careful in interpreting Eq. (1). Now

$$\frac{q_s}{(2\pi)^2} G_0 = \int_{\chi_0}^{\chi_p} d\mu[\Phi_2 - \tfrac{1}{2}(\chi\Phi_2)'] + \int_{\chi_s}^{\chi_0} d\mu[(\Phi_2 - \Phi_1) - \tfrac{1}{2}\{\chi(\Phi_2 - \Phi_1)\}'] = \int_{\chi_s}^{\chi_p} d\mu\,[\Phi_\infty - \tfrac{1}{2}(\chi\Phi_\infty)'] = \frac{q_s}{(2\pi)^2} G_\infty. \tag{2}$$

Therefore, to the extent that $d\mu(\chi)$ is arbitrary, G represents an extended class of global integrals preserved by all ideal motions and those nonideal motions that are permitted by the type of reconnection process considered here. K, which corresponds to the simplest choice $w(\chi) = 1$, is only one member of this class, but the only one independent of q_s. It may be shown easily that G is gauge invariant.

We suggest now the following variant of the thought experiment of Kruskal and Kulsrud.[1] We imagine a slightly nonideal plasma contained in a toroidal vessel with perfectly conducting walls. The plasma is turbulent with tearing modes of different m and n. The existence of fine-scale tearing destroys all invariants to some extent, except $K = \int_{V_0} d^3r \vec{A}\cdot\vec{B}/2$. On a short time scale, however, the $m = 1, n = 1$ mode may be assumed to be least affected by other modes, and the "first moment" with respect to $\chi\ (\equiv \Psi - \Phi)$, $K_1 = \int_{V_0} d^3r\, \chi\vec{A}\cdot\vec{B}/2$ the best conserved of all invariants other than Φ_p and K. Since the two latter invariants are, respectively, linear and quadratic in the fluxes, the choice of the functional K_1, cubic in the fluxes, as the next best invariant seems eminently reasonable. We shall see that this choice is vindicated by agreement with experimental observations.

We seek, therefore, minima of $W = \int_{V_0} d^3r\, B^2/2$ subject to the global invariants $K = \int_{V_0} d^3r\, \chi\vec{A}\cdot\vec{B}/2$. We must have $\delta W - \lambda\delta K - \lambda_1\,\delta K_1 = 0$, where λ and λ_1 are Lagrange multipliers. With the boundary conditions $\hat{n}\cdot\vec{B} = 0$, $\delta\Psi = 0$, $\delta\Phi = 0$ at the conducting wall, we obtain the Euler-Lagrange equation $\vec{J} = \lambda[1 + (\Psi - \Phi)/\Phi_p]\vec{B}$, where we have chosen $3\lambda_1/2 = \lambda\Phi_p$ in order that the toroidal (and poloidal) current density vanish at the wall. This is an experimental boundary condition violated by Taylor's theory.[4]

For a straight cylinder we use cylindrical polar coordinates (r, θ, z) and assume that equilibrium quantities depend only on $r (B_r = 0)$. We have defined $\vec{\bar{B}} \equiv \vec{B}/2\Phi_p$, $\bar{\Psi} \equiv \Psi/2\Phi_p$, and $\bar{\Phi} \equiv \Phi/2\Phi_p$. The boundary conditions are $(a \equiv 1)$ $\bar{B}_\theta(0) = 0$, $\bar{\Psi}(1) = 0$, $\bar{\Phi}(0) = 0$, and $\bar{\Phi}(1) = \frac{1}{2}$. This two-point boundary-value problem has been solved numerically by a shooting procedure. The numerical results are qualitatively similar for aspect ratios from 10 to 1, and we have reported the results for $R/a = 5$. For any given $\lambda \in (-\infty, +\infty)$ there are two distinct branches, which we have broadly classified as "pinchlike" (P) and "tokamaklike" (T). In Fig. 3, we compare the predictions of our theory with recent experimental measurements of the F-θ trajectory $[F = \bar{B}_z(1),\ \theta = \bar{B}_\theta(1)]$ during self-reversal in ZT-40.[9]

Figure 4 shows a plot of $V \equiv 2R^{-1}W/(2\pi\Phi_p)^2$ vs $R^{-1}K/(2\pi\Phi_p)^2$ for the solutions. The point 0, which corresponds to $|\lambda| = \infty$, is a branch point from which four solutions emerge. For a given value of $K/(2\pi\Phi_p)^2$ (V s/toroidal flux), the plasma should prefer the lower-energy states indicated by the solid lines. In fact, if experimental conditions should drive the plasma to the higher-energy states indicated by the dashed lines, instabilities would immediately set in, forcing the plasma to lower-energy states. A preliminary examination of the stability of these states indicates stable windows of operation for $\theta \lesssim 0.2$ and $1.6 \lesssim \theta$

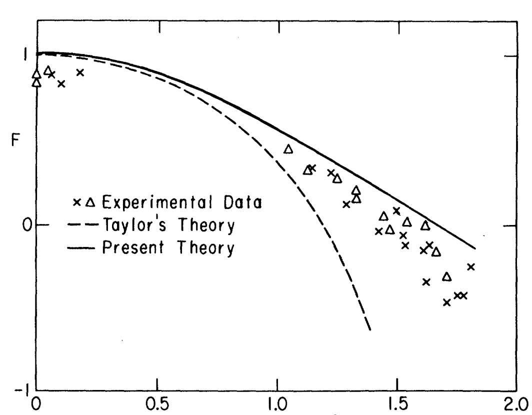

FIG. 3. Comparison of theoretical predictions with F-θ plot from two typical shots in ZT-40.

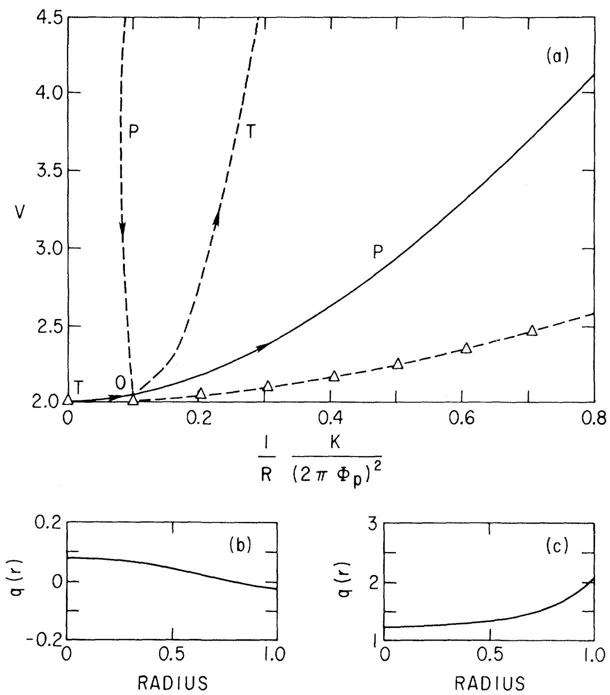

FIG. 4. (a) Energy of equilibria in present theory compared with energy of Taylor states (marked by Δ). Arrows indicate direction of increasing λ. Labels P and T distinguish pinchlike and tokamaklike equilibria. Dashed lines indicate unstable equilibrium (energy stationary, but not minimum). (b) Typical q profile on the stable P branch. (c) Typical q profile on the stable T branch.

for $R/a = 5$. The first window is "tokamaklike" and the latter "pinchlike." Figures 4(b) and 4(c) show typical stable q profiles. The equilibrium equations admit an expansion in powers of inverse aspect ratio. The leading-order solutions are

$$\overline{B}_z(r) \simeq 1, \;\; \overline{B}_\theta(r) \simeq \sum_{n=1}^{\infty} \frac{8\lambda}{\lambda_n{}^2(\lambda_n{}^2 + 2R\lambda)} \frac{J_1(\lambda_n r)}{J_1(\lambda_n)}, \qquad (3)$$

where λ_n corresponds to the solutions of $J_0(\lambda_n) = 0$. Equation (3) agrees very well with the numerical solutions for the "tokamaklike" branch.

An important aspect of this theory is that it allows a natural extension to equilibria with nonzero pressure gradients, unlike the equilibria in Taylor's theory which are force-free even in the presence of finite pressure. Details will be reported elsewhere.

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