course, with the electromagnetic field switched off) belong here.

On the other hand, the real slices of the complexified $S(a, b, c/m)$ structures, completed by *all* their limiting cases (contractions), are likely to cover all those Harrison metrics[9] which admit two commuting Killing vectors. The classification of the Harrison metrics being related to specific separability criteria and having thus no direct geometrical interpretation, and the involved nature of the problem of finding out all real slices of complexified $S(a, b, c/m)$ structures—together with their contractions—make the verification of the conjecture expressed above an interesting open problem.

It may be also conjectured that the $S(a, b, c/m)$ real structure taken as germinal (seed) solution within the new Kinnersley-Chitre generating technique,[10] most easily by use of the method of Ernst and Hauser,[11] can induce solutions more general than those of the T-S family (see, for example, Refs. 12 and 13), at least some of them being of physical interest.

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[1]The details of the process which led to g_4 as an integral of $G_{\mu\nu} = 0$, including a study of the conformal curvature and the singularities, will be published elsewhere.

[2]M. D. Kruskal, Phys. Rev. 119, 1743 (1960).

[3]E. Kasner, Trans. Am. Math. Soc. 27, 101 (1921).

[4]K. Rózga, Rep. Math. Phys. 11, 197 (1977).

[5]*Perspectives in Geometry and Relativity*, edited by B. Hoffman (Indiana Univ. Press, Bloomington, 1966), p. 176.

[6]F. Ernst, Phys. Rev. 167, 1175 (1968), and 168, 1415 (1968).

[7]W. Kinnersley, J. Math. Phys. (N. Y.) 14, 651 (1973).

[8]L. Witten and P. Esposito, in *Relativity, Fields, Strings and Gravity*, edited by C. Aragone (Univ. Simon Bolivar Press, Carácas, 1975).

[9]B. K. Harrison, Phys. Rev. 116, 1285 (1959).

[10]W. Kinnersley and D. M. Chitre, J. Math. Phys. (N. Y.) 18, 1538 (1977), and 19, 1926, 2037 (1978).

[11]I. Hauser and F. Ernst, J. Math. Phys. (N. Y.) 19, 1316 (1978), and 20, 1041 (1979), and Phys. Rev. D 20, 362, 1783 (1979), and Illinois Institute of Technology Report No. PORS-IIT-19, 20, 1980 (unpublished).

[12]A. Tomimatsu and H. Sato, Phys. Rev. Lett. 29, 1344 (1972), and Prog. Theor. Phys. 50, 95 (1973).

[13]M. Yamazaki, J. Math. Phys. (N. Y.) 18, 2502 (1977), and 19, 1376 (1978), and Prog. Theor. Phys. 57, 1951 (1977).

New Class of Soluble-Model Boltzmann Equations

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The Boltzmann equation considered by Bobylev, and Krook and Wu (BKW) is rewritten in the form of a stochastic equation, similar in form to the kinetic equations of Tjon and Wu, and Ernst. A new class of models, which reflects the transformation of the BKW model to other dimensionalities, is constructed, and its equilibrium distributions, nonequilibrium solutions (corresponding to the BKW mode), and H theorems are found.

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Following the discovery by Krook and Wu[1] and by Bobylev[2] of an exact solution to the Boltzmann equation for a spatially uniform system, other authors, including Tjon and Wu[3] and Ernst and co-workers,[4-7] have found related kinetic models that also allow exact solutions. These latter models are defined by kinetic equations in the form of a stochastic equation which may be written

$$\frac{\partial F(x,t)}{\partial t} = \int_0^\infty dy\, F(y,t) \int_0^\infty dz\, F(z,t) P(y,z;x) - F(x,t) \int_0^\infty dy\, F(y,t) \int_0^\infty dz\, P(x,y;z), \qquad (1)$$

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where $F(x,t)$ is the energy distribution function, $x=v^2/2$ is the kinetic energy, v is the velocity, and t is time. F does not depend upon direction or position because the system is assumed to be isotropic and spatially uniform. $P(y,z;x)$ is the transition or collision probability for the collision $y+z\to x$ [the other particle having energy $(y+z-x)$], and characterizes the particular model. For example, the Tjon-Wu (TW) model corresponds to (1) with P given by

$$P^{\rm TW}(y,z;x)=(y+z)^{-1},\quad x<y+z. \tag{2}$$

Note that for all models P must be zero when $x>y+z$ in order to conserve energy. The class of models discussed by Ernst[4] corresponds to

$$P^{{\rm E}(m)}(y,z;x)=\frac{\Gamma(2m)}{\Gamma(m)^2}\frac{[x(y+z-x)]^{m-1}}{(y+z)^{2m-1}},\quad x<(y+z), \tag{3}$$

and Ernst and Hendriks[6] also considered the model $P(y,z;x)=1$. Note that, in general, P has the symmetry

$$P(y,z;x)=P(z,y;x)=P(y,z;y+z-x) \tag{4}$$

and that consequently (1) has two constants of motion, M_0 and M_1 (mass and energy), where

$$M_n(t)\equiv\int_0^\infty x^nF(x,t)\,dx. \tag{5}$$

In this discussion, we will always assume that the units are chosen such that $M_0=1$, while the value of M_1 will vary with the model.

The work of Bobylev and of Krook and Wu (BKW) concerned the Boltzmann equation itself, describing an isotropic system of Maxwell-like molecules in which the scattering cross section σ is inversely proportional to the relative velocity, and independent of the scattering angle Ω:

$$\frac{\partial f(v)}{\partial t}=\frac{1}{4\pi}\int d\Omega\int d^3w[f(v')f(w')-f(v)f(w)]. \tag{6}$$

Here $f(v)=f(v,t)$ is the velocity distribution function, related to F by $F(x,t)=4\pi vf(v,t)$, $x=v^2/2$. In terms of F, the nonequilibrium solution to (6) found by BKW is given by

$$F(x,t)=\frac{2(x)^{1/2}e^{-x/K}}{\pi^{1/2}K^{5/2}}\left(\frac{5K-3}{2}+x\frac{1-K}{K}\right), \tag{7}$$

where $K\equiv1-\exp(-t/6)$. The TW equation, although much simpler in appearance, is intimately related to the BKW equation, such that solutions of one can be transformed directly into solutions to the other.[8]

I have found that the BKW equation, (6), may also be written in the form of (1), and that P is given by

$$P^{\rm BKW}(y,z;x)=(yz)^{-1/2}\times\begin{cases}\arcsin[x/(y+z)]^{1/2}, & 0<x<y\\ \arcsin[y/(y+z)]^{1/2}, & y<x<z\\ \arcsin[1-x/(y+z)]^{1/2}, & z<x<y+z\end{cases} \tag{8}$$

(for $y<z$). A derivation of (8) will be given in a later paper,[9] while a method of verifying it will be indicated below. Note that (1) with (8) is a much simpler equation for $P(x,t)$ than (6), since the former contains two one-dimensional integrals, while the latter involves two- and three-dimensional integrals, and implicitly, the scattering dynamics. The BKW model can be stated in (almost) as simple a form as the other models above.

$P^{\rm BKW}$ clearly contains much more structure than the P of the other models. It depends upon the energies y and z of each particle entering the collision, rather than upon just their sum $(y+z)$ as in (3)–(5). $P^{\rm BKW}$ is discontinuous (in the derivative) at $x=y$ and z, and is constant in between these two values. In this interval, particles leave the collisions with equally probable energy. The integral of $P^{\rm BKW}$ with respect to x equals unity,

$$\int_0^\infty P(y,z;x)\,dx=1, \tag{9}$$

the same as in models (2) and (3), implying that the loss term of (1) is simply $F(x,t)$.

Inspired by some of the above properties of $P^{\rm BKW}$, I construct a model in which P is defined by

$$P^*(y,z;x)=\begin{cases}0, & 0<x<y\\ |z-y|^{-1}, & y<x<z.\\ 0, & x>z\end{cases} \tag{10}$$

In this model, the energy of the outgoing particles is restricted to lie in the interval bounded by the incoming particles' energy. Using (1), we find that the moments satisfy the same equation,

$$\frac{dM_n^*}{dt}+M_n^*=\frac{1}{n+1}\sum_{i=0}^{n}M_i^*M_{n-i}^* \tag{11}$$

as the *renormalized* moments of the BKW model,[1] $[2^n n!/(2n+1)!]M_n^{\mathrm{BKW}}$, and also satisfied by the renormalized TW moments, $M_n^{\mathrm{TW}}/n!$ By virtue of this relation between moments, I have derived (using a Laplace-transform technique) the following relation between the distribution functions of these models:

$$F^{\mathrm{TW}}(x,t)=\int_0^\infty F^*(z,t)z^{-1}e^{-x/z}dz \tag{12}$$

and

$$F^{\mathrm{BKW}}(x,t)=\frac{2\sqrt{x}}{\sqrt{\pi}}\int_0^\infty F^*(z,t)z^{-3/2}e^{-x/z}dz. \tag{13}$$

Thus, any solution $F^*(x,t)$ of the model defined by (10) can be transformed to a solution of the TW and BKW models. These formulas are consistent with the relation between F^{BKW} and F^{TW} that has been given by Barnsley and Turchetti.[8]

It turns out that the transformation (13) has been put forth by Alexanian[10] in a different context, in which (13) reflects the representation of F^{BKW} as a superposition of Maxwell-Boltzmann distributions of temperature z, $2(x/\pi)^{1/2}z^{-3/2}e^{-x/z}$. This provides a physical interpretation of $F^*(z,t)$ as the "temperature" distribution of the system. Alexanian also derived a kinetic equation for F^*, Eqs. (2)–(3) in Ref. 10, and although his equation is of a form much different from (1) and (10) above, they can be shown to be equivalent.

The equilibrium distribution of this model is given by

$$F_{\mathrm{eq}}^*(x)=\delta(x-1). \tag{14}$$

Evidently, the restriction that P^* imposes on the outgoing particles causes the distribution function to sharpen and eventually turn into a δ function, at which time all particles have the same energy or speed. By virtue of (13), the BKW mode, (7), translates into the nonequilibrium solution (also given by Alexanian)

$$F^*(x,t)=\delta(x-K)-(1-K)(\partial/\partial x)\delta(x-K) \tag{15}$$

for this model. Note that (15) is negative at $x=K^-$ for all finite time. Even though this model exhibits singular behavior, it might prove useful for numerical studies, similar to those undertaken on the TW model,[3] to investigate the significance of the BKW mode in the general approach to equilibrium. The advantage of this model for numerical studies over the other models is that the distribution function here does not spread in energy space as time increases, thus eliminating the need to impose numerical cutoffs.

For the present purposes, the major significance of the above model is that it can be used to generate an infinite class of new models. We observed that (12) and (13) can both be obtained from the general equation

$$F^{(m)}(x,t)=\frac{x^{m-1}}{\Gamma(m)}\int_0^\infty F^*(z,t)z^{-m}e^{-x/z}dz \tag{16}$$

with $m=1$ and $m=\frac{3}{2}$, respectively. Considering (16) for all $m>0$ defines the new class. Using (14) I find that the general equilibrium distribution is given by

$$F_{\mathrm{eq}}^{(m)}(x)=[x^{m-1}/\Gamma(m)]e^{-x}, \tag{17}$$

and using (15) I find the generalization of the BKW mode:

$$F^{(m)}(x,t)=\frac{x^{m-1}e^{-x/K}}{\Gamma(m)K^{m+1}}\left[(K+mK-m)+x\frac{1-K}{K}\right]. \tag{18}$$

The moments of this class are related to that of (10) by

$$M_n^{(m)}=[\Gamma(n+m)/\Gamma(m)]M_n^*. \tag{19}$$

To complete the description of these models, the corresponding $P^{(m)}$ must be found. The moment equation for $M_n^{(m)}$ follows from (19) and (11), and assuming $F^{(m)}(x,t)$ satisfies an equation in the form of (1), with $P^{(m)}$ satisfying (9), a general moment equation can also be written. Equating these two, I find that $P^{(m)}$ must satisfy

$$\int_0^{y+z}x^nP^{(m)}(y,z;x)=\frac{\Gamma(m)\Gamma(m+n)}{n+1}\sum_{i=0}^{n}\frac{y^iz^{n-i}}{\Gamma(i+m)\Gamma(n-i+m)} \tag{20}$$

[which is consistent with (9)]. By various techniques, including the use of a Laplace transform, I have found that the $P^{(m)}$ are given by the following expresssions. The $P^{(m)}$ are generally of the form

$$P^{(m)}(y,z;x)=\frac{(y+z)^{2m-3}}{(yz)^{m-1}}\begin{cases} q^{(m)}[x/(y+z)], & 0<x<y \\ q^{(m)}[y/(y+z)], & y<x<z \\ q^{(m)}[1-x/(y+z)], & z<x<y+z \end{cases} \tag{21}$$

for $y<z$, in which $P^{(m)}$ is constant for $y<x<z$, and $q^{(m)}$ depends only upon $x/(y+z)$. Note that (2) and (8) are in this form, with $q^{(1)}=1$ and $q^{(3/2)}=\arcsin\sqrt{u}$. In general, the $q^{(m)}$ are given by

$$q^{(m)}(u)=(m-1)\int_0^u[u(1-u)]^{m-2}du \tag{22}$$

(for $m>1$), or

$$q^{(m)}(u)=\frac{u^{m-1}\Gamma(m)}{\pi}\sum_{i=0}^{\infty}\frac{(-u)^i\Gamma(i-m+2)\sin\pi(m-1+i)}{i!\,\Gamma(m-1+i)} \tag{23}$$

for m nonintegral. Explicit expressions for $m=\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3 are given in Table I. For example, $P^{(2)}$ is in the form of a trapezoid. $P^{(1/2)}$ has the unusual behavior that it goes to ∞ at $x=0$ and $y+z$; the increased production of particles at zero energy leads to an equilibrium distribution $F_{\mathrm{eq}}^{(1/2)}(x)=e^{-x}/(\pi x)^{1/2}$ which goes to ∞ at $x=0$. It can be verified directly that (21)–(23) satisfy (20). Incidentally, this serves as a way to verify that (8) truly represents the BKW model, for in verifying (20) for $m=\frac{3}{2}$, one proves that (8) implies the Krook-Wu moment equation, (11) and (19).

Finally, an H theorem can be derived for these models. With use of the fact that

$$\tilde{P}(y,z;x)\equiv(yz)^{m-1}P^{(m)}(y,z;x) \tag{24}$$

has the inverse collision symmetry

$$\tilde{P}(y,\xi-y;x)=\tilde{P}(x,\xi-x;y) \tag{25}$$

one can readily show that the function

$$H(t)=\int_0^{\infty}F^{(m)}(x,t)\ln\left(\frac{F^{(m)}(x,t)}{x^{m-1}}\right)dx \tag{26}$$

satisfies $dH/dt\leq 0$, thus proving the monotonic approach to the equilibrium distribution (17).

This new class of models is very closely related to the class considered by Ernst. According to (22), $q^{(m)}$ is the incomplete β-function while

TABLE I. Some explicit expressions for $q^{(m)}(u)$.

m	$q^{(m)}(u)$
1/2	$(1-2u)/[u(1-u)]^{1/2}$
1	1
3/2	$\arcsin\sqrt{u}$
2	u
5/2	$(3/8)\{\arcsin u^{1/2}-(1-2u)[u(1-u)]^{1/2}\}$
3	$u^2-(2/3)u^3$

Ernst's probability, (3), is the related β distribution. Of course, Ernst's model has none of the discontinuities exhibited by (21). The equilibrium distribution of Ernst's class is identical to (17) and the H theorem goes through the same way, since $(yz)^{m-1}P^{\mathrm{E}(m)}$ also shows the symmetry of (25). The generalization of the BKW mode is identical to (18), except that now $K=1-\exp(-\lambda t)$, where $\lambda=m/[2(2m+2)]$. When $m=2$, the two classes coincide and both become the TW model.

Ernst derived his class of models as a representation of "diffuse" scattering in a $2m$-dimensional system of Maxwell-like molecules. Note that the energy distribution in a d-dimensional system is related to the velocity distribution by $F(x,t)\propto v^{d-2}f(v,t)$ and therefore $F_{\mathrm{eq}}(x)\propto x^{d/2-1}e^{-x}$, in agreement with the equilibrium distribution (17). In no way can a member of either class of models $P^{(m)}$ or $P^{\mathrm{E}(m)}$ represent a three-dimensional model when $m\neq\frac{3}{2}$, in the sense that the model follows from a three-dimensional Boltzmann equation, since the latter always gives, for any expression for σ, the equilibrium energy distribution $F_{\mathrm{eq}}(x)\propto x^{1/2}e^{-x}$. The present class can also be thought of as representing a $2m$-dimensional system, giving the TW model in two dimensions and the BKW model in three. The $q^{(m)}$ listed in Table I therefore represent the dimensionalities 1 through 6.

It should be noted that in a very recent paper, Futcher and Hoare[11] have discussed a class of kinetic models (describing a two-step collision process) in which P is given by *products* of incomplete β functions. Like the class of models discussed here, their model is nondiffusive in that $P(y,z;x)$ depends upon the individual values of y and z rather than just $y+z$.

Further areas of research include the study of models for which (9) does not hold and instead the integral of P with respect to x depends upon y and z. The model studied by Ernst and Hendriks[6] is one such model. For these so-called non-Maxwell models, the loss term of (1) will not be simply $F(x,t)$ as in the models we have discussed. In principle, the isotropic Boltzmann equation can always be written in the form of (1) with P represented by a certain integral of σ. It would be useful if a practical form of this expression for P can be found.

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[1]M. Krook and T. T. Wu, Phys. Fluids 20, 1589 (1977), and Phys. Rev. Lett. 36, 1107 (1976).

[2]A. V. Bobylev, Dokl. Acad. Nauk. SSSR 20, 1296 (1975) [Sov. Phys. Dokl. 20, 822 (1976)].

[3]J. A. Tjon and T. T. Wu, Phys. Rev. A 19, 883 (1979); J. A. Tjon, Phys. Lett. 70A, 369 (1979).

[4]M. H. Ernst, Phys. Lett. 69A, 390 (1979).

[5]M. H. Ernst, in Proceedings of the Conference on Mathematical Problems in the Kinetic Theory of Gases, Oberwolfach, West Germany, 20–26 May 1979, edited by H. Neunzert (to be published).

[6]M. H. Ernst and E. M. Hendriks, Phys. Lett. 70A, 183 (1979).

[7]E. Futcher, M. R. Hoare, E. M. Hendriks, and M. H. Ernst, to be published.

[8]M. Barnsley and G. Turchetti, Phys. Lett. 72A, 417 (1979).

[9]R. M. Ziff, to be published.

[10]M. Alexanian, Phys. Lett. 74A, 1 (1979).

[11]E. J. Futcher and M. R. Hoare, Phys. Lett. 75A, 443 (1980).