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Undamped Second-Sound Waves in a ^{3}He-^{4}He Mixture Heated from Below

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By using a thermodynamic analysis, it is shown that a horizontal layer of the ^{3}He-^{4}He superfluid mixture heated from below is unstable with respect to oscillatory convection. The neutral oscillations of this overstability are undamped standing second-sound waves. Estimates show the possibility to observe the predicted effect experimentally in the vicinity of the tricritical point of the ^{3}He-^{4}He mixture.

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Perturbations of a noncompressible superfluid ^{3}He-^{4}He mixture lead to the appearance of second-sound waves which decay rapidly. The situation, however, changes if the system is heated from below. As shown in this note, the second-sound wave becomes an undamped standing wave. This wave is a neutral oscillation of the convective overstability.

A ^{3}He-^{4}He superfluid mixture is an interesting object for the study of the convective instability because of the high attainable temperature resolution and the low heat transport.[1] Moreover, the thermodynamic and kinetic properties of this mixture vary by several orders of magnitude within the ^{3}He-^{4}He phase diagram between the λ line and the coexistence curve.

An analysis of stationary and oscillatory instabilities in the ^{3}He-^{4}He superfluid mixture by the standard methods will be published elsewhere.[2,3] Here, we discuss only one branch of the convective instability in this system which is, in fact, the undamped second-sound wave. It is well known that in the ^{3}He-^{4}He superfluid mixture in contrast to pure He II a temperature gradient can exist in equilibrium. This temperature gradient leads to a corresponding concentration gradient, so that the superfluid ^{4}He component moves to the warm boundary and, therefore, the light ^{3}He atoms are concentrated near the cold boundary. This unusual distribution of concentration with height is a result of the superfluidity. A similar concentration distribution occurs in binary mixtures with a large abnormal thermodiffusional (Soret) effect, $k_T > 0$.[4] Such systems are unstable with respect to stationary convection when they are heated from above and with respect to oscillatory convection when heated from below.[5,6] The oscillatory instability, in this case, appears as a result of an interaction between the diffusion and thermal modes. The physical nature of the oscillatory instability in the ^{3}He-^{4}He superfluid mixture is absolutely different. Here the oscillatory instability manifests itself as a second-sound wave mode. For this mode to be undamped, the rate of supply of energy by the buoyancy force has to balance the rate of wave dissipation. This qualitative statement can be formulated in a quantitative form by using Chandrasekhar's thermodynamic principle.[7]

For the sake of simplicity, we consider a horizontal layer with free boundaries a distance l apart. Following the usual linear stability analysis, we linearize the two-fluid hydrodynamic equations for the superfluid mixture.[8] We express the small perturbations of the vertical component of the normal velocity, V_{nz}', the entropy per one gram of ^{3}He, σ', and the chemical potential of the ^{4}He atoms, μ_4', in the form

$$[V_{nz}'; \sigma'; \mu_4'] = (V, \sigma, \xi) e^{i\vec{k}\cdot\vec{r}} \cos(\pi z). \qquad (1)$$

The dependence on coordinates in Eq. (1) is chosen in accordance with accepted free boundary conditions.

The linearized convection equations can be transformed to the form[9]

$$-\frac{\partial V}{\partial t}(\pi^2+k^2)=(\pi^2+k^2)V+k^2\left[N_{\rm Ra}\sigma+L\left(\frac{l}{l_0}\right)^3\xi\right],$$

$$\frac{\partial^2\xi}{\partial t^2}+rP_T^{-1}(\pi^2+k^2)\frac{\partial\xi}{\partial t}+\omega_0^2(\pi^2+k^2)\xi=-\frac{N_{\rm Ra}}{L}\left(\frac{l_0}{l}\right)^3 s(\pi^2+k^2)\frac{\partial\sigma}{\partial t}, \quad (2)$$

$$P_T\frac{\partial\sigma}{\partial t}+(\pi^2+k^2)\sigma=-V-\frac{Ld}{N_{\rm Ra}}\left(\frac{l}{l_0}\right)^3(\pi^2+k^2)\xi.$$

Here we use the symbols

$$k^2=k_x^2+k_y^2,$$

$$N_{\rm Ra}=\frac{1}{\rho_n}\left(\frac{\partial\rho}{\partial\sigma}\right)_{P,\mu_4}\frac{\rho_n g l^4}{\eta\kappa}\frac{d\sigma_0}{dz},$$

$$\kappa=\frac{\chi_{\rm eff}}{\rho c T}\left(\frac{\partial T}{\partial\sigma}\right)_{P,\mu_4},$$

$$l_0^3=\eta\kappa/g\rho_n,\quad P_T=\eta/\rho_n\kappa,\quad P_c=\eta/\rho_n D,$$

$$L=\frac{gl}{\rho_n}\left(\frac{\partial\rho}{\partial\mu_4}\right)_{P,\sigma},\quad r=\left(\frac{n_1}{n}\frac{P_T}{P_c}-\frac{ad}{n}\right), \quad (3)$$

$$s=\frac{a}{nP_T}\left(\frac{P_T}{P_c}\frac{a_1}{a}-1\right),\quad \omega_0^2=-\frac{1}{nP_TL}\frac{\rho_s}{\rho_n}\left(\frac{l}{l_0}\right)^3,$$

$$a=\frac{\rho_n}{c}\frac{(\partial c/\partial\sigma)_{P,\mu_4}}{(\partial\rho/\partial\sigma)_{P,\mu_4}},$$

$$a_1=a\left[1+\frac{k_T}{T}\frac{(\partial T/\partial\sigma)_{P,\mu_4}}{(\partial c/\partial\sigma)_{P,\mu_4}}\right],\quad n=\frac{\rho_n}{c}\frac{(\partial c/\partial\mu_4)_{P,\sigma}}{(\partial\rho/\partial\mu_4)_{P,\sigma}},$$

$$n_1=n\left[1+\frac{k_T}{T}\frac{(\partial T/\partial\mu_4)_{P,\sigma}}{(\partial c/\partial\mu_4)_{P,\sigma}}\right],$$

$$d=\frac{(\partial T/\partial\mu_4)_{P,\sigma}}{(\partial T/\partial\sigma)_{P,\mu_4}}\frac{(\partial\rho/\partial\sigma)_{P,\mu_4}}{(\partial\rho/\partial\mu_4)_{P,\sigma}}.$$

The variables V, σ, ξ, t, and z in Eq. (2) are scaled by χl^{-1}, $(d/dz)l$, gl, $\rho_n l^2/\eta$, and l, respectively.

Let us note that two assumptions are made in the hydrodynamic equations (2). Firstly, we assume that dissipation of the superfluid motion is small, $\rho_n(\zeta_4-\rho\zeta_3)/\eta<1$. Secondly, it is assumed that dissipation of a normal motion is also small, $(l_0/l)^3<1$. Then in the Navier-Stokes equation there is used $\mathrm{div}\vec{V}_n=0$.[2] To find an oscillatory instability, we look for a solution of the problem in the form

$$V=v\cos(\omega t). \quad (4)$$

On substituting (4) into the second and the third equations of (2), one obtains

$$\sigma=A\cos(\omega t)+B\sin(\omega t),$$
$$\xi=F\cos(\omega t)+E\sin(\omega t), \quad (5)$$

where the coefficients A, B, F, and E are functions of the parameters defined in Eq. (3).

According to Chandrasekhar,[7] oscillatory convection will appear when the oscillating rate of change of kinetic energy per unit volume $\dot{E}_0$ is balanced by the viscous dissipation rate $\dot{E}_v$ (with a 90° phase lag with respect to the change of kinetic energy) and the rate of energy production by the buoyancy force $\dot{E}_g$, i.e.,

$$\dot{E}_0=\dot{E}_v+\dot{E}_g. \quad (6)$$

All the functions in Eq. (6) can be found from the first equation in (2), and we obtain

$$\dot{E}_0=\int_{-1/2}^{1/2}V_{nz}\frac{\partial}{\partial t}\Delta V_{nz}\,dz=\tfrac{1}{2}(\pi^2+k^2)v^2\omega\cos(\omega t)\sin(\omega t), \quad (7a)$$

$$\dot{E}_v=\int_{-1/2}^{1/2}V_{nz}\Delta^2V_{nz}\,dz=\tfrac{1}{2}(\pi^2+k^2)v^2\cos^2(\omega t), \quad (7b)$$

$$\dot{E}_g=\int_{-1/2}^{1/2}V_{nz}\left(\frac{d}{dz}\mathrm{div}-\Delta\right)\left[N_{\rm Ra}\sigma'+L\left(\frac{l}{l_0}\right)^3\mu_4'\right]dz$$
$$=\tfrac{1}{2}k^2v\cos(\omega t)\left\{\left[AN_{\rm Ra}+FL\left(\frac{l}{l_0}\right)^3\right]\cos(\omega t)+\left[BN_{\rm Ra}+EL\left(\frac{l}{l_0}\right)^3\right]\sin(\omega t)\right\}. \quad (7c)$$

The $\cos^2(\omega t)$ term in $\dot{E}_g$ in Eq. (7c) is offset by the viscous dissipation Eq. (7b), while the $\cos(\omega t)\times\sin(\omega t)$ term balances an oscillation of the kinetic energy of the viscous fluid described by Eq. (7a). Thus one obtains two equations for the critical Rayleigh number, $N_{\rm Ra}^{\rm osc}$, which determines the onset of convection and the frequency of the overstable motion ω_*. Restricting our attention here only to the second-sound wave branch of the instability, we find from Eqs. (7a) and (7c) that[10]

$$\omega_*^2=\omega_0^2(\pi^2+k^2). \quad (8)$$

Let us note that in the case considered here $(\pi^2+k^2)^2/P_T\omega_0^2\ll 1$, since we neglect the diffusion and the thermal modes. From Eqs. (7b) and (7c), one obtains the Rayleigh number which determines the onset of the oscillatory instability:

$$N_{\mathrm{Ra}}{}^{\mathrm{osc}} \equiv N_{\mathrm{Ra}} L\left(\frac{l_0}{l}\right)^3 \frac{\rho_n}{\rho_s} n\left[1+\frac{aP_c - a_1P_T}{n_1P_T - adP_c}\right]. \qquad (9)$$

The result Eq. (9) is confirmed, of course, by the general linear stability analyses.[3]

After minimizing Eq. (9) in k, one obtains the condition for the onset of overstability

$$N_{\mathrm{Ra}}{}^{\mathrm{osc}} = 4\pi^2 \text{ at } k_0 = \pi, \qquad (10)$$

and the neutral oscillation frequency

$$\omega_*{}^2 = 2\pi^2\omega_0{}^2. \qquad (11)$$

Since $\omega_0{}^2 > 0$, it is clear that $N_{\mathrm{Ra}}{}^{\mathrm{osc}} < 0$, i.e., the oscillatory instability in the horizontal layer of the superfluid ^{3}He-^{4}He mixture appears when it is heated from below.

Let us show now that ω_0 is the frequency of the standing second-sound waves. The velocity of the second-sound wave in an incompressible fluid mixture in the variables P, σ, and μ_4 is equal to

$$u_2{}^2 = -\frac{\rho_s}{\rho_n} c\left(\frac{\partial\mu_4}{\partial c}\right)_{P,\sigma}. \qquad (12)$$

Thus the characteristic frequency is given by

$$\omega_2{}^2 = u_2{}^2k^2 = -\frac{\pi^2}{l^2}\frac{\rho_s}{\rho_n} c\left(\frac{\partial\mu_4}{\partial c}\right)_{P,\sigma}. \qquad (13)$$

Equation (13) is the same as the dimensional form of ω_0 in Eq. (3). Now I would like to emphasize here that the criterion (9) is proportional to the square of frequencies ratio

$$N_{\mathrm{Ra}}{}^{\mathrm{osc}} \sim -(\omega_g/\omega_2)^2,$$

where $\omega_g{}^2 = (g/\rho_n)(\partial\rho/\partial\sigma)_{P,\mu_4} d\sigma_0/dz$ is the internal-gravity-wave frequency.[9] This expression is very different from usual Rayleigh criterion that consists in the relation between the buoyancy force and stabilizing dissipation factor. It follows from more detailed analysis[3] that the effect considered can only be observed in the vicinity of the tricritical point (T_t, x_t).

Using the well-known singularities of the thermodynamic and kinetic properties near the tricritical point of the ^{3}He-^{4}He mixture,[11] one can easily find the following expressions for the criteria of the onset of the oscillatory instability and for the appropriate neutral frequency:

$$N_{\mathrm{Ra}}{}^{\mathrm{osc}} = 10^{-4} l^2 (dT_0/dz)\epsilon_t{}^{-2}, \quad \omega^{*2} = 4\times 10^8 l^{-2}\epsilon_t, \qquad (14)$$

with $\epsilon_t = 1 - T/T_t$. For $l = 5$ cm and $\epsilon_t = 10^{-4}$, one obtains for the critical temperature gradient and the neutral frequency

$$dT_0/dz \simeq 10^{-4} \text{ K/cm}, \quad \omega_* \simeq 100 \text{ Hz}. \qquad (15)$$

Therefore, it should be possible to observe the effect predicted above in the vicinity of the tricritical point of the ^{3}He-^{4}He mixture.[12]

After completion of this work, experimental results were published[13] which show the possibility of stationary convective instability in dilute ^{3}He-^{4}He mixtures.

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[10]Notice that for $\omega = 0$ the solution of (7b) and (7c) determines the onset of stationary convection $N_{\mathrm{Ra}}{}^{\mathrm{st}} \equiv N_{\mathrm{Ra}}$, $N_{\mathrm{Ra}}{}^{\mathrm{st}} = (\pi^2 + k^2)^3/k^2$ and after minimization $N_{\mathrm{Ra}}{}^{\mathrm{st}} = 27\pi^4/4$ at $k_0{}^2 = \frac{1}{2}\pi^2$ (Ref. 2). This criterion was first written in Ref. 9.

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