Loop-Space Formulation of Gauge Theories

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Some intricate properties of the integrability conditions of the loop-space chiral equations, which do not have their correspondence in the ordinary chiral equations, are pointed out.

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Recently the loop formulation[1] of gauge theories has been given much attention. It has the nice properties of being the minimal necessary object for a gauge theory, having simple gauge-transformation properties[2] and being the spinlike quantity for the lattice field theory.[3] Of particular interest, besides the derivation of the stringlike equations,[4] is the realization that the Yang-Mills equations can be formulated in the loop space as a formal analogy of the classical chiral equations.[5,6] Thus it has been suggested that the inverse-scattering equations can be formulated for these loop-space chiral equations and the existence of infinite numbers of conservation laws will follow just like the ordinary chiral equation.[7] This raised the hope that the loop-space chiral equations were also a totally integrable system and, therefore, possibly lead to the full solution of the Yang-Mills equations. However, here we want to discuss some intricate properties of the integrability conditions of the loop-space chiral equations, which do not have their correspondence in the ordinary chiral[8] equations.

First we discuss the integrability conditions of the nonlocal currents in two possible different situations. In the first case, the "generating" functions are functionals of the loop alone. We show that the integrability conditions are not satisfied and higher-order conserved nonlocal currents do not exist. In the second case, the "generating" functions are functionals of the loop as well as a parameter. The integrability conditions at a restricted point of the parameter are satisfied; however, there is an infinite fold of arbitrariness. These points are then demonstrated with an explicit example. Afterwards we make connection of the loop-space chiral equation with the formulation of the inverse-scattering equations in the loop space and again show similar conclusions. The points elaborated here indicate that additional guiding principles are needed in order to make unique connections between the loop-space chiral equations and the infinite conserved nonlocal currents or the inverse-scattering formulation.

Chiral Fields in Loop Space.—Let us consider the phase factor along a closed loop $l = x^\mu(s)$,

$$\Phi_{02s10} = \psi(l) = P\exp(i\oint A_\mu dx_\mu). \tag{1}$$

The functional differentiation of the loop phase factor is defined as the change in $\psi(l)$, as l changes to l', which is infinitesimally deformed from l at s,

$$\frac{\delta\psi(l)}{\delta x^\mu(s)} = \frac{\psi(l') - \psi(l)}{ds\,dx^\mu(x)} = \Phi_{02s} f_{\lambda\mu}[x(s)]\frac{dx_\lambda(x)}{ds}\Phi_{s10}. \tag{2}$$

It is just the parallel transported "normal flux" per unit area that went through the deformed area. Define the loop-space gauge potential

$$\mathfrak{F}_\mu(l,s) \equiv \psi(l)^{-1}\delta\psi(l)/\delta x^\mu(s) = \Phi_{01s} f_{\lambda\mu}[x(s)]\dot{x}_\lambda(s)\Phi_{s10}. \tag{3}$$

Functionally differentiating it again, and after some work, one can show[4,5]

$$\frac{\delta\mathfrak{F}_\mu(l,s)}{\delta x^\nu(s')} - \frac{\delta\mathfrak{F}_\nu(l,s')}{\delta x^\mu(s)} + [\mathfrak{F}_\mu(l,s), \mathfrak{F}_\nu(l,s')] = 0, \tag{4}$$

which just gives the projected Bianchi identity

$$\{\mathfrak{D}_\mu f_{\nu\lambda}[x(s)]\}\dot{x}_\lambda + \{\mathfrak{D}_\nu f_{\lambda\mu}[x(s)]\}\dot{x}_\lambda + \{\mathfrak{D}_\lambda f_{\mu\nu}[x(s)]\}\dot{x}_\lambda = 0, \tag{4a}$$

and

$$\delta \mathfrak{F}_\mu(l,s)/\delta x_\mu(s)=0, \tag{5}$$

which gives the projected Yang-Mills equation

$$\{\mathfrak{D}_\mu f_{\mu\nu}[x(s)]\}\dot{x}_\nu(s)=0, \tag{5a}$$

where $\mathfrak{D}_\mu$ denotes covariant differentation. The geometric meaning of the loop-space Eq. (4) is that the loop phase factor arrived from an initial loop to a given final loop is independent of the different volumes swapped out by the intermediate loops.

Integrability of the "Generating" Functions.—Equation (5) is like a continuity equation, and so we try to follow the standard procedure and identify the first current as follows (here we specialize to the case in two dimensions, though the conclusion is general):

$$V_\mu^{(1)}(l,s)\equiv \mathfrak{F}_\mu(l,s)=\epsilon_{\mu\nu}\delta\chi^{(1)}/\delta x^\nu(s). \tag{6}$$

This satisfies Eq. (5), but the question is whether Eq. (5) provides the sufficient conditions for the integration of $\chi^{(1)}$ from Eq. (6). We shall discuss separately the following two possible cases.

Case (1): $\chi^{(1)}$ is a functional of the loop above, i.e., Eq. (6) reads

$$\mathfrak{F}_\mu(l,s)=\epsilon_{\mu\nu}\delta\chi^{(1)}(l)/\delta x^\nu(s). \tag{6'}$$

Just as in the finite-dimensional case, the integrability condition of $\chi^{(1)}(l)$ is

$$\frac{\delta^2\chi^{(1)}(l)}{\delta x^\nu(s')\,\delta x^\mu(s)}-\frac{\delta^2\chi^{(1)}(l)}{\delta x^\mu(s)\,\delta x^\nu(s')}=0. \tag{7}$$

From $\delta\chi^{(1)}(l)/\delta x^\nu(s)=-\epsilon_{\nu\mu}\mathfrak{F}_\mu(l,s)$, Eq. (7) gives

$$\epsilon_{\mu\alpha}\delta\mathfrak{F}_\alpha(l,s)/\delta x^\nu(s')=\epsilon_{\nu\alpha}\delta\mathfrak{F}_\alpha(l,s')/\delta x^\mu(s); \tag{8}$$

for $\mu=\nu=1$,

$$\delta\mathfrak{F}_2(l,s)/\delta x^1(s')=\delta\mathfrak{F}_2(l,s')/\delta x^1(s), \tag{8a}$$

which is false, unless $s'\to s$; for $\mu=1$ and $\nu=2$,

$$\delta\mathfrak{F}_2(l,s)/\delta x^2(s')=-\delta\mathfrak{F}_1(l,s')/\delta x^1(s), \tag{8b}$$

which becomes Eq. (5) only in the limit $s'\to s$. Therefore, we see that higher conserved currents cannot be constructed by this procedure.

Case (2): $\chi^{(1)}$ is not only a functional of the loop but also a parameter s. Now Eq. (6) becomes

$$\mathfrak{F}_\mu(l,s)=\lim_{s'\to s}\epsilon_{\mu\nu}\frac{\delta\chi^{(1)}(l,s)}{\delta x^\nu(s')}=\epsilon_{\mu\nu}\frac{\delta\chi^{(1)}(l,s)}{\delta x^\nu(s)}. \tag{6''}$$

Thus the integrability condition of Eq. (6) becomes

$$\frac{\delta^2\chi^{(1)}(l,s)}{\delta x^\nu(s)\,\delta x^\mu(s)}-\frac{\delta^2\chi^{(1)}(l,s)}{\delta x^\mu(s)\,\delta x^\nu(s)}=0. \tag{7'}$$

Notice that all parameters coincide at a point s. Then from $\delta\chi^{(1)}(l,s)/\delta x^\nu(s)=\epsilon_{\nu\mu}\mathfrak{F}_\mu(l,s)$ the integrability condition becomes Eq. (8) with $s'\to s$. Thus the equations of motion Eq. (5) do provide integrability of $\chi^{(1)}(l,s)$ from (6′). However, the peculiar situation here is that Eq. (6′) constrains $\chi(l,s)$ only when the parameter s' of $\delta x_\mu(s')$ coincides with s of $\chi(l,s)$, thus it does not sufficiently constrain $\chi(l,s)$ and there are infinitely many $\chi(l,s)$'s that can satisfy Eq. (6″). This is a manifestation that additional information is needed in order to integrate uniquely the system from one point of the loop to the other.

Since $\chi^{(1)}(l,s)$ can be constructed, now we can follow the standard iterative procedure to construct the nth current from the $(n-1)$th current,

$$V_\mu^{(n)}(l,s)=\lim_{s'\to s}\left[\frac{\delta}{\delta x_\mu(s')}+\mathfrak{F}_\mu(l,s)\right]\chi^{(n-1)}(l,s), \tag{9}$$

where $\chi^{(n-1)}$ satisfies

$$V_\mu^{(n-1)}(l,s)=\epsilon_{\mu\nu}\delta\chi^{(n-1)}(l,s)/\delta x^\nu(s).$$

Using the original equations of motion Eqs. (4) and (5), one can easily show the $V_\mu^{(n)}(l,s)$'s are conserved, i.e.,

$$\delta V_\mu^{(n)}(l,s)/\delta x_\mu(s)=0. \tag{10}$$

Notice here that the arbitrariness in $\chi^{(n-1)}(l,s)$ is reflected directly in the next current $V_\mu^{(n)}(l,s)$.

Actually both cases can be demonstrated explicitly from the solution of the Yang-Mills equation in two dimensions. In R^2 we have the following solution for the gauge potentials of the Yang-Mills equations:

$$b_1=-x_2c,\quad b_2=0,\quad \text{and then } f_{12}=c, \tag{11}$$

where c is an element of g. Of course, we assume $c\neq 0$. It is easily seen that Eq. (11) is the general solution of the Yang-Mills equations in R^2. From Eq. (3) we have

$$\mathfrak{F}_1(l,s)=c\,dx_2(s)/ds,\qquad \mathfrak{F}_2(l,s)=-c\,dx_1(s)/ds. \tag{12}$$

The analogy of Eq. (6) is

$$\delta\chi/\delta x^1(s)=\mathfrak{F}_2(l,s)=-c\,dx_1(s)/ds,\qquad \delta\chi/\delta x^2(s)=-\mathfrak{F}_1(l,s)=-c\,dx_2(s)/ds. \tag{13}$$

In case (1), $\chi=\chi(l)$ a functional of loop alone, clearly from Eq. (13),

$$\delta^2\chi(l)/\delta x^i(s')\,\delta x^i(s)=-c\dot{\delta}(s'-s)\neq\delta^2\chi(l)/\delta x^i(s)\,\delta x^i(s')=-c\dot{\delta}(s-s'), \tag{14}$$

where $i=1$ or 2, since $\mathring{\delta}(s'-s)$ is odd under interchange of s and s'. But in case (2), if we allow $\chi'=\chi(l,s)$, a functional of both the loop and a parameter, then Eq. (13) has an infinite number of solutions. One of them has been constructed by Polyakov,[9]

$$\chi(l,s) = \frac{c}{\pi}\int dt\,du \frac{(s-t)(s-u)}{[(s-t)^2+(s-u)^2]^2}x_\mu(t)x_\mu(u), \quad (15)$$

and

$$\frac{\delta\chi(l,s)}{\delta x_\mu(t)} = \frac{2c}{\pi}\int du \frac{(s-t)(s-u)}{[(s-t)^2+(s-u)^2]^2}x_\mu(u). \quad (16)$$

Using the fact $\mathring{\delta}(x)=\lim_{\epsilon\to 0}[-2\epsilon x/\pi(x^2+\epsilon^2)^2]$, so that $\lim_{s\to t}\{(s-t)(s-u)/[(s-t)^2+(s-u)^2]^2\} = -\frac{1}{2}\pi\mathring{\delta}(s-u)$, we obtain

$$\lim_{t\to s}\frac{\delta\chi(l,s)}{\delta x^\mu(t)} = \frac{\delta\chi(l,s)}{\delta x^\mu(s)} = -c\frac{dx_\mu(s)}{ds},$$

which is precisely Eq. (13). However, this is only one of the infinite ways of constructing the solution. We can replace $(s-t)(s-u)/[(s-t)^2+(s-u)^2]^2$ by any function, e.g., $\{[(s-t)/(s-u)^3]\exp[-(s-u)^2/(s-t)^2]+(t\leftrightarrow u)\}$, which is $-\frac{1}{2}\pi\mathring{\delta}(s-u)$ in the limit $t\to s$. This is a consequence of the fact that Eqs. (6″) do not constrain $\chi(l,s)$ when s' of $\delta x^\mu(s')$ and s of $\chi(l,s)$ are different.

Loop-Space Inverse-Scattering Equations and Their Integrability.—Following the inverse-scattering equations in the ordinary space

$$\left(\frac{d}{dx^\mu}+A_\mu(x)-\gamma\epsilon_{\mu\nu}\frac{d}{dx^\nu}\right)\varphi(x,\gamma)=0, \quad (17)$$

where γ is an arbitrary parameter, we like to see what a similar formulation for the loop space will imply.

Case (1): In this case the higher-order conservation laws do not exist, and so the inverse-scattering equations cannot be constructed from them with use of the standard method.[8] However, to demonstrate a point, following the spirit of Ref. 5, we construct one in analogy to Eq. (17),

$$\left(\frac{\delta}{\delta x^\mu(s)}+\mathfrak{F}_\mu(l,s)-\gamma\epsilon_{\mu\nu}\frac{\delta}{\delta x^\nu(s)}\right)\Phi(l,\gamma)=0. \quad (18)$$

Note that $\Phi(l,\gamma)$ is a functional of loop and the parameter γ. What are the conditions for the integration of $\Phi(l,\gamma)$? Equation (18) can be rewritten as

$$\frac{\delta}{\delta x_\mu(s)}\Phi(l,\gamma) = -\frac{1}{1+\gamma^2}[\mathfrak{F}_\mu(l,s)+\gamma\epsilon_{\mu\alpha}\mathfrak{F}_\alpha(l,s)]\Phi(l,\gamma). \quad (19)$$

Requiring, for arbitrary γ,

$$\frac{\delta^2\Phi(l,\gamma)}{\delta x^\mu(s)\,\delta x^\nu(s')}-\frac{\delta^2\Phi(l,\gamma)}{\delta x^\nu(s')\,\delta x^\mu(s)}=0, \quad (20)$$

one obtains the following conditions for arbitrary s and s': For the γ^{-3} term,

$$\epsilon_{\mu\alpha}\frac{\delta\mathfrak{F}_\alpha(l,s)}{\delta x^\nu(s')}-\epsilon_{\nu\beta}\frac{\delta\mathfrak{F}_\beta(l,s')}{\delta x^\mu(s)}=0, \quad (21)$$

which for $\mu=1$ and $\nu=1$ becomes

$$\frac{\delta\mathfrak{F}_2(l,s)}{\delta x^1(s')}-\frac{\delta\mathfrak{F}_2(l,s')}{\delta x^1(s)}=0, \quad (21a)$$

and which for $\mu=1$ and $\nu=2$ becomes

$$\frac{\delta\mathfrak{F}_2(l,s)}{\delta x^2(s')}+\frac{\delta\mathfrak{F}_1(l,s')}{\delta x^1(s)}=0; \quad (21b)$$

for the γ^{-2} term,

$$-\frac{\delta\mathfrak{F}_\mu(l,s')}{\delta x^\nu(s)}+\frac{\delta\mathfrak{F}_\nu(l,s)}{\delta x^\mu(s')}+[\epsilon_{\mu\alpha}\mathfrak{F}_\alpha(l,s'),\epsilon_{\nu\beta}\mathfrak{F}_\beta(l,s)]=0; \quad (22)$$

for the γ^{-1} term,

$$-\epsilon_{\mu\alpha}\frac{\delta\mathfrak{F}_\alpha(l,s')}{\delta x^\nu(s)}+\epsilon_{\nu\beta}\frac{\delta\mathfrak{F}_\beta(l,s)}{\delta x^\mu(s')}+[\epsilon_{\mu\alpha}\mathfrak{F}_\alpha(l,s'),\mathfrak{F}_\nu(l,s)]+[\mathfrak{F}_\mu(l,s'),\epsilon_{\nu\beta}\mathfrak{F}_\beta(l,s)]=0; \quad (23)$$

and for the γ^0 term,

$$\frac{\delta\mathfrak{F}_\mu(l,s')}{\delta x^\nu(s)}-\frac{\delta\mathfrak{F}_\nu(l,s)}{\delta x^\mu(s')}+[\mathfrak{F}_\mu(l,s'),\mathfrak{F}_\nu(l,s)]=0. \quad (24)$$

We see that they require much more than the loop-space chiral Eqs. (4) and (5) for integrability.

Case (2): In this case infinite number of conserved currents satisfying Eq. (5) can be constructed, so that we have

$$V_\mu^{(n)}(l,s)=\epsilon_{\mu\nu}\frac{\delta}{\delta x_\nu(s)}\chi^{(n)}(l,s)=\left(\frac{\delta}{\delta x_\mu(s)}+\mathfrak{F}_\mu(l,s)\right)\chi^{(n-1)}(l,s). \tag{25}$$

Multiplying Eq. (25) by L^n and summing over all n, and then defining $\varphi(l,s,L)\equiv\sum_{n=0}L^n\chi^{(n)}(l,s)$, we obtain the inverse-scattering equation for $\varphi(l,s,L)$, which is in the form of Eq. (18) with $\gamma=L^{-1}$,

$$\lim_{s'\to s}\left(\frac{\delta}{\delta x^\mu(s')}+\mathfrak{F}_\mu(l,s)-L^{-1}\epsilon_{\mu\nu}\frac{\delta}{\delta x_\nu(s')}\right)\Phi(l,s,L)=0. \tag{18$'$}$$

The integrability conditions in the limit $s'\to s$ are just from Eqs. (21) to (24) with $s'\to s$, which imply Eqs. (4) and (5), the equations of motion.

In conclusion, the above discussions indicate that the loop-space chiral equations are not a totally integrable system in the ordinary sense. The loop-space chiral equations do not provide enough information for the integration of loop-space currents from one point of the loop to another in a unique way.[10] However, in spite of such difficulties, the observation that the Yang-Mills equations give the loop-space chiral equations is such a beautiful one that, with further insight, it is bound to lead to new understanding of the gauge theories.

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[9]A. M. Polyakov, private communication. We thank him for showing us this solution, which helps to demonstrate the situation clearly.

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