New Derivations of the Quadrupole Formulas and Balance Equations for Gravitationally Bound Systems

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A new derivation of the quadrupole formulas and balance equations of general relativity for gravitationally bound systems is given which overcomes objections raised to previous derivations.

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The use of the so-called quadrupole formulas of general relativity for the calculation of the rates at which energy and angular momentum is radiated by gravitationally bound systems such as the binary pulsar PSR 1913+16 has been criticized by a number of authors.[1] Furthermore, several of these authors have derived expressions for these rates that differ from the quadrupole expressions by factors of the order of unity.[2] In addition the derivations of energy and angular-momentum balance equations needed to calculate secular changes in the orbit elements of such systems have also been questioned.

Until recently there did not appear to be any possibility of testing the predictions of the quadrupole formulas—they all gave loss rates when applied to known astrophysical systems that were far too small to be observable. The discovery of PSR 1913+16 appears to make possible for the first time a test of these predictions. Its orbital period has been observed to be decreasing at the rate of the order of 10^{-12} sec/sec. The most recent observations[3] give, in fact, a value for the period change of (1.04 ± 0.13) times the quadrupole prediction. If this period change is due entirely to losses by gravitational radiation, then this result is now sufficiently accurate to rule out all predictions other than the quadrupole prediction itself. It is therefore important to decide which of these predictions is a valid consequence of general relativity if one wishes to use the observed period change to test this theory. In this Letter I will outline a derivation of the quadrupole formulas and balance equations for gravitationally bound systems which I believe is free of the objections raised to previous such derivations.

The method we shall use to obtain an approximate expression for the period change of a binary system makes use of the Landau-Lifshitz complex[4] $\theta^{\mu\nu}$ which satisfies, as a consequence of the field equations of general relativity, the conservation law[5]

$$\theta^{\mu\nu}{}_{,\nu} = 0. \qquad (1)$$

One method, used by Peters,[6] to derive the quadrupole formula from Eq. (1) is to integrate it over a $t = t_0$ hypersurface bounded by a sphere S whose radius r is allowed to approach infinity. When this is done and use is made of Gauss's theorem one obtains

$$(d/dt)\int_v \theta^{00}\, dV + \int \theta^{0i}\, n_i dS = 0. \qquad (2)$$

The problem with using this equation as it stands is that the fields appearing in $\theta^{\mu\nu}$ are retarded fields which depend on the retarded coordinates of the sources.

In order to evaluate the integrals appearing in (2) it is necessary to expand the retarded quantities in a Taylor series in terms of their instantaneous values at the time t_0. Since the effective sources which come from the nonlinear terms in the field equations of general relativity are nonlocal, such a procedure results in a series in which the first few terms are finite but in which all the subsequent terms diverge. This is not to say that the integrals in Eq. (2) diverge—only that such an expansion leads to divergent results.

One can avoid these divergent terms by integrating Eq. (1) over a four-volume bounded by three hypersurfaces Σ_1, Σ_2, and S. The hypersurface S is defined by the condition $r = R$ in the limit $R \to \infty$. One can show that such a choice for S is convenient but not necessary; any spacelike hypersurface that is sufficiently far removed from the sources will yield the same results. The hypersurfaces Σ_1 and Σ_2 are chosen to coincide in the wave zone with two future-directed null cones and only in the near zone are they taken to be the hypersurfaces $t = t_1$ and $t = t_2$. Performing the indicated integration and using Gauss's theorem leads to the balance equations

$$\int_{\Sigma_2} \theta^{\mu\nu}\, d\sigma_\nu - \int_{\Sigma_1} \theta^{\mu\nu}\, d\sigma_\nu + \int_S \theta^{\mu\nu}\, d\sigma_\nu = 0. \qquad (3)$$

The virtue of this procedure is that all of the integrals appearing in Eq. (3) can, in first approximation, be related to the values of the coordinates of the sources for times lying between t_1 and t_2 without the appearance of divergent integrals. In what follows I shall refer to Eq. (3) with $\mu=0$ as the energy balance equation. In a similar manner one can obtain an angular-momentum balance equation

$$\int_{\Sigma_2} M^{ij\nu}\, d\sigma_\nu - \int_{\Sigma_1} M^{ij\nu}\, d\sigma_\nu + \int_S M^{ij\nu} d\sigma_\nu = 0, \qquad (4)$$

where

$$M^{ij\nu} = x^i\theta^{j\nu} - x^j\theta^{i\nu}, \quad i, j = 1, 2, 3. \qquad (5)$$

To proceed with the derivation we must obtain an approximate expression for the fields appearing in the integrals over S, Σ_1, and Σ_2 in Eqs. (3) and (4) in terms of the coordinates of the sources. These expressions are obtained by solving the field equations of general relativity by some approximate means. One approximate method that has been used in the past for this purpose involves linearizing the field equations and then solving the resulting linear equations by imposing deDonder coordinate conditions and an outgoing radiation condition. The resulting solution is a standard retarded integral of the stress-energy-momentum tensor $T^{\mu\nu}$ of the sources. The rationale for this procedure is that the fields are weak and hence the nonlinear terms in the field equations can be neglected in lowest order. The resulting retarded integrals of $T^{\mu\nu}$ are then evaluated in terms of a sum of integrals over its instantaneous values which in turn are evaluated by means of a series of integrations by parts.[7]

As long as one is dealing with nongravitationally bound sources one can probably justify this procedure although Madore[8] has argued that even in this case the linearized solution is not an approximation to any exact solution of the field equations. The real difficulty arises, however, when one considers gravitationally bound sources such as a double-star system. In this case the gravitational stresses arising from the nonlinear terms in the field equations are of the same order of magnitude as the mechanical stresses (recall the virial theorem) and so cannot be neglected in the field equations even in the lowest order of approximation. As a consequence the procedure outlined above for obtaining the gravitational field in terms of the source coordinates again leads to divergent terms in higher orders of the approximation. Furthermore the neglect of surface terms that arise from integrations by parts in the finite terms must also be justified anew.

One of the main sources of these difficulties in the past has been the use of regular perturbation theory in which one expands the gravitational field in a power series in some small parameter; often the difficulties are compounded by taking a dimensional quantity such as $1/c$ in the so-called slow-motion approximation to be the parameter. It is well known[9] that the use of regular perturbation methods often leads to nonuniformities and singularities that are not intrinsic to the problem but rather are an artifact of these methods. Fortunately perturbation methods have been developed that avoid these difficulties. Burke[10] was one of the first ones to recognize the need for such so-called singular perturbation methods in general relativity with his use of the method of matched asymptotic expansions. In this work I have found it necessary to make use not only of this method but also Lighthill's method of stretched coordinates and the method of multiple time scales.

The method of matching is required whenever a small parameter multiplies the highest derivative in a differential equation as it does for slowly moving systems in general relativity. In this case the small parameter, which we designate by ϵ, is the light travel time across the system divided by its period or, what is the same thing, the size of the system divided by the wavelength of the radiation emitted by it. In the case of the binary pulsar PSR 1319+16 ϵ has the value 6.5×10^{-4}.

To apply the method of matching one recognizes two regions, an inner or near zone and an outer or wave zone. One solves the field equations in these two zones by some approximate means and then matches the inner expansion of the outer solution to the outer expansion of the inner solution to determine the arbitrary functions appearing in the two solutions. In both zones one uses a time coordinate $t^* = \epsilon t$, where t is measured in units of the light travel time across the system. In addition one uses in the outer zone a radial coordinate $r^* = \epsilon r$, where r is measured in units of the size of the system. In our application of this method we take, in both zones, the basic field variables to be the deviations $\gamma^{\mu\nu}$ of the metric density $(-g)^{1/2}g^{\mu\nu}$ from a Minkowsky metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, i.e.,

$$\gamma^{\mu\nu} = (-g)^{1/2}g^{\mu\nu} - \eta^{\mu\nu}. \qquad (6)$$

We further assume that, in the outer zone, $\gamma^{\mu\nu}$ depends on t^* through its dependence on $u^* = \epsilon u$, where u is the future-directed null coordinate and satisfies the equation

$$(-g)^{1/2} g^{\mu\nu} u_{,\mu} u_{,\nu} = 0. \tag{7}$$

The use of the null coordinate u as described here is required to avoid the appearance of nonuniformities that would otherwise arise in the outer solution in higher orders of approximation as $\ln r^*$ factors.[11] Since $\gamma^{\mu\nu}$ is itself to be determined by an approximation procedure it follows that u must also be so determined. This use of u as an effective independent variable is in essence the method of strained coordinates.

In order to solve the field equations of general relativity it is necessary to impose coordinate conditions on the $\gamma^{\mu\nu}$. Because of the simplifications their use entails, it has been customary to employ the so-called deDonder conditions $\gamma^{\mu\nu}{}_{,\nu} = 0$ for this purpose. However, as Fock[12] first pointed out, these conditions also lead to nonuniformities in the outer solution through the appearance of additional $\ln r^*$ factors. I have found[11] that we can avoid these nonuniformities and still achieve significant simplification of the field equations if we require, instead of the deDonder conditions,

$$\dot{\gamma}^{\mu\nu}{}_{,\nu} = -\tfrac{1}{4} \dot{\gamma}^{\alpha\beta} \dot{\gamma}_{\alpha\beta} \gamma^{\mu\nu} u_{,\nu}, \tag{8}$$

where the "dot" denotes differentiation with respect to u. Fortunately, to the order of accuracy needed to determine orbit-element changes these conditions reduce to the deDonder conditions.

In addition to coordinate conditions it is also necessary to impose some kind of a radiation condition on solutions of the outer problem. For this purpose we shall require that in the far wave zone, where $r^* \gg 1$, $\gamma^{\mu\nu}$ can be approximated by an asymptotic series in inverse powers of r^*:

$$\gamma^{\mu\nu} \sim \sum_{m=1}^{\infty} (r^*)^{-m} {}_m\gamma^{\mu\nu}(u^*, \vec{n}), \tag{9}$$

where $\vec{n}$ is an outward radial unit vector. We also require that in this zone the coordinate t^* can also be approximated by an asymptotic series of the form

$$t^* \sim u^* + r^* + {}_0b \ln r^* + \sum_{m=1}^{\infty} (r^*)^{-m} {}_m b(u^*, \vec{n}), \tag{10}$$

where ${}_0b$ is a constant. [Unless ${}_0b$ is a constant the expansions (9) and (10) will not satisfy the field equations.] For spatially compact sources such as we are dealing with we impose two conditions on the coefficients appearing in the expansions (9) and (10). To eliminate the contribution of source free waves to $\gamma^{\mu\nu}$ we require that these coefficients tend to zero if the source strength tends to zero. To eliminate noncausal solutions we require that these coefficients be independent of u in the space-time region $u < u_0$ if the source is stationary in this region.

When use is made of the above conditions and when the expansions (9) and (10) are inserted into the field equations, it is found that all of the coefficients appearing in these expansions are determined in terms of the one set of coefficients ${}_1\gamma^{\mu\nu}$ which appear in the expansion (9) and a number of constants of integration. It is further found that when the surface S is allowed to approach future null infinity in Eqs. (3) and (4) the integrands of the integrals over S are given as a sum of terms which are quadratic in the ${}_1\gamma^{ij}$ and ${}_1\dot{\gamma}^{ij}$.

The last steps in the derivation involve the determination of ${}_1\gamma^{\mu\nu}$ in terms of the source variables. For this purpose one uses the method of matched asymptotic expansions with the matching proceeding along the lines of Burke's derivation. The only essential difference is that the functions F_M that appear in Burke's lowest-order (in ϵ) outer solutions are now functions of u^* rather than $t^* - r^*$. The net result is that in lowest order ${}_1\gamma^{ij}$ is given by

$${}_1\gamma^{ij} \cong 2\ddot{Q}^{ij}(u) + O(\epsilon^2), \tag{11}$$

where Q^{ij} is the reduced quadrupole moment given by

$$Q^{ij}(t) = \int \rho(t, \vec{x}) \{x^i x^j - \tfrac{1}{3} r^2 \delta^{ij}\}\, d^3x, \tag{12}$$

where $\rho(t, \vec{x})$ is the mass density of the source. When these expressions for ${}_1\gamma^{ij}$ are substituted into the integrals over S in the balance equations one obtains the standard quadrupole results for the rates of energy and angular momentum loss given, for example, in Misner, Thorne, and Wheeler.[13]

The final step in the derivation involves the evaluation of the integrals over the two hypersurfaces Σ_1 and Σ_2 in the balance equations. If the system were not radiating these two integrals would be equal to each other and, in lowest approximation, to the Newtonian expressions for the energy and angular momentum of the system. However, because the system is radiating, the orbit elements will in general change in time with the consequence that the integrals over Σ_1 and Σ_2

will no longer be equal to each other nor can they be expressed simply in terms of these orbit elements. Fortunately in the case of the binary pulsar the time scale for these changes is of the order of ϵ^{-5} times the period of the system. As a consequence one can make use of the method of multiple time scales to evaluate these integrals.

In this method one assumes that the orbit elements are functions of a "slow" time $\epsilon^5 t$ while the other motion variables are functions of a "fast" time t. In lowest order of approximation one can then neglect the slow-time variation of the orbit elements in evaluating the integrals over Σ_1 and Σ_2. The result is that they are again in lowest order of approximation equal to the Newtonian expressions for the energy and angular momentum of the system. They differ slightly in value from each other, however, because the orbit elements appearing in these expressions have slightly different values on Σ_1 and Σ_2. The difference between the two integrals can therefore be expressed in terms of the rates of change of the orbit elements and these in turn can be related to the integrals over S in the balance equations (3) and (4). In this way one obtains expressions for the rates of change of the semimajor axis and eccentricity of the Newtonian orbits that are equal to the expressions (5.6) and 5.7) given by Peters and used by Taylor in the analysis of his observations.

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[1]See, for example, J. Ehlers, A. Rosenblum, J. N. Goldberg, and P. Havas, Astrophys. J. 208, L77 (1976); F. Cooperstock and D. Hobill, Phys. Rev. D 20, 2995 (1979).

[2]N. Hu, to be published, obtains a factor of 0.735; A. Rosenblum, Phys. Rev. Lett. 41, 1003 (1978), obtains a factor of 2.5.

[3]J. H. Taylor, private communication.

[4]L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).

[5]Greek indices take values from 0 to 3, Latin indices take values 1 to 3. The Einstein summation convention is employed and ordinary differentiation is denoted by a comma: $\partial/\partial x^\mu \equiv {}_{,\mu}$.

[6]P. C. Peters, Phys. Rev. 136, B1224 (1964).

[7]Landau and Lifshitz, Ref. 4.

[8]J. Madore, Ann. Inst. Henri Poincaré 12, 285 (1970).

[9]For a discussion of the kinds of difficulties one encounters with the use of regular perturbation methods and a description of the singular perturbation methods referred to in this Letter see, for example, A. Neyfeh, *Perturbation Methods* (Wiley, New York, 1973).

[10]W. Burke, J. Math. Phys. (N.Y.) 12, 401 (1970).

[11]J. L. Anderson, in *Isolated Gravitating Systems in General Relativity*, edited by J. Ehlers (North-Holland, Amsterdam, 1979).

[12]V. Fock, *The Theory of Space Time and Gravitation* (Pergamon, New York, 1959).

[13]C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

Equilibrium Polymerization as a Critical Phenomenon

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Equilibrium polymerization can be described by the $n \rightarrow 0$ limit of the n-vector model of magnetism in a small magnetic field. Nonclassical critical effects are predicted. The earlier theory of Tobolsky and Eisenberg is a mean-field approximation to the present theory. An application is made to the polymerization in sulfur.

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A great variety of organic and inorganic compounds can polymerize to form linear and nonlinear polymers.[1] In many cases, polymerization proceeds under conditions of equilibrium between monomer and polymer,[2] and interesting transition phenomena can occur.[3]