Vanishing One-Loop β Function in Gauged $N > 4$ Supergravity

S. M. Christensen
Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106

and

M. J. Duff
Physics Department, Imperial College, London SW7 2AZ, United Kingdom

and

G. W. Gibbons
Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106, and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom[a]

and

M. Roček
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom
(Received 4 April 1980)

In O(N) extended supergravity theories, the usual arguments for one-loop finiteness (modulo topological terms) cease to apply when the internal symmetry is gauged because of the appearance of a cosmological constant related to the gauge coupling e. For $N \leq 4$, we find that infinite renormalizations are required. Remarkably, the particle content of theories with $N > 4$ results in a cancellation of these infinities, implying, in particular a vanishing one-loop $\beta(e)$ function.

PACS numbers: 11.10 Np, 04.60+n, 11.10 Gh, 11.30.Pb

Although the renormalizability properties of quantum gravity and supergravity have received considerable attention over the last few years, almost all these investigations have confined their attention to theories with vanishing cosmological constant, Λ. In a recent paper,[1] however, explicit results for one-loop counterterms and anomalous scaling behavior were given both for pure gravity and for gravity plus matter fields of spin 0, $\frac{1}{2}$, and 1, in the presence of a cosmological constant. Related work may be found in Gibbons and Perry.[2] The present note summarizes the calculations of a forthcoming publication[3] in which these techniques are generalized to spin-$\frac{3}{2}$ fields and hence to supergravity,[4] with dramatic results for the O(N) extended models.[5]

These developments may appear as something of a luxury: If ordinary quantum gravity is nonrenormalizable without a cosmological constant, it is not likely to become so with the additional complication of a nonvanishing Λ. However, such arguments require drastic revision in extended supergravity where the gauging of the O(N) symmetry[6] requires a (huge) cosmological constant $\Lambda = -6e^2/\kappa^2$ and gravitino mass parameter m, with $\Lambda = -3m^2$. (e is the gauge coupling constant, and $\kappa^2 = 8\pi G$ where G is Newton's constant.) Thus it is plausible that the ultraviolet behavior of these models at higher loops, by virtue of their extra local symmetry, may even be an improvement over theories without a cosmological constant. Moreover, the strong empirical evidence in favor of a vanishing cosmological constant may be only an apparent discrepancy between theory and experiment if, as suggested in Hawking's "space-time foam,"[7] one reinterprets Λ as a measure of the average small-scale curvature of space-time.

Let us first recall the pure gravity results of Refs. 1 and 2. If, at the classical level, we take the Einstein action

$$S = -(1/2\kappa^2)\int d^4x\, g^{1/2}(R - 2\Lambda), \tag{1}$$

then, using the background field method,[8] the one-loop counterterms will be a linear combination of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $R_{\mu\nu}R^{\mu\nu}$, R^2, ΛR, and Λ^2 but with gauge-dependent coefficients. Gauge invariance is achieved by use of the field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Alternatively, terms which vanish with the field equations may be removed by gauge-dependent field redefinition.[9] Either way, the resulting

© 1980 The American Physical Society

counterterm ΔS may then be written

$$\Delta S=-(1/\epsilon)\gamma, \qquad (2)$$

where $\epsilon=n-4$ is the dimensional regularization parameter. γ is given by

$$\gamma=A\chi+B\delta, \qquad (3)$$

where A and B are numerical coefficients and where

$$\chi\equiv(1/32\pi^2)\int d^4x\, g^{1/2}\, {}^*P_{\mu\nu\rho\sigma}{}^*R^{\mu\nu\rho\sigma}, \qquad (4)$$

$$\delta\equiv\frac{1}{12\pi^2}\int d^4x\, g^{1/2}\,\Lambda^2=-\frac{\kappa^2\Lambda}{12\pi^2}S. \qquad (5)$$

The star denotes the duality operation, and

$$g^{1/2}\,{}^*R_{\mu\nu\rho\sigma}{}^*R^{\mu\nu\rho\sigma}=g^{1/2}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2)$$

is a total divergence which is sometimes discarded. Its integral over all space, however, yields the Euler number χ, a topological invariant which takes on integer values in spaces with nontrivial topology. The explicit calculations of Ref. 1 yield $A=106/45$ and $B=-87/10$. Thus, in contrast to the case $\Lambda=0$,[10] pure gravity with a Λ term is no longer one-loop "finite" (in the nontopological sense) because $B\neq 0$.

One may now repeat the exercise for simple supergravity with a gravitino mass term[11]:

$$S=\int d^4x\,(\det e_\mu{}^a)\left[-\frac{1}{2\kappa^2}R+\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma+m\bar\psi_\mu\sigma^{\mu\nu}\psi_\nu+\tfrac{1}{3}(s^2+p^2-A^\mu A_\mu)+\frac{2m}{\kappa}s\right]. \qquad (6)$$

Elimination of the auxiliary field s yields a cosmological constant $\Lambda=-3m^2$. The one-loop counterterms will now be given by the appropriate supersymmetric completion[12] of those encountered in pure gravity, i.e., ΔS is again given by $-\epsilon^{-1}(A\chi+B\delta)$ with $\delta=-\kappa^2\Lambda(12\pi^2)^{-1}S$ (on shell) but where S is now given by Eq. (6). The topological invariant χ, on the other hand, acquires no extra terms.[13] The coefficients A and B will now receive contributions from both the graviton and the gravitino (with its appropriate mass parameter). Explicit calculations in Ref. 3 yield $A=\frac{41}{24}$ and $B=-\frac{77}{12}$ and, once again in contrast to the case $\Lambda=0=m$,[14] simple supergravity is no longer one-loop finite.

We now combine these results with those of Ref. 1 for spins 1, $\frac{1}{2}$, and 0, and apply them to the extended O(N) theories with gauged internal symmetry. The coefficient B now takes on a new significance: By supersymmetry it also determines the renormalization of the gauge coupling constant e.

The supersymmetric completion of δ now contains the spin-1 gauge field contribution $e^2\,\mathrm{Tr}F_{\mu\nu}\times F^{\mu\nu}$. Note that this arises from two different sources: In addition to the usual charge renormalization effects, there will also be one-loop counterterms of the form $\kappa^2R\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$. With use of the field equations $R=4\Lambda+\ldots$ with $\kappa^2\Lambda=-6e^2$ this is converted into an extra $e^2\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ term.[15]

Before the displaying of our results, some qualifications are required. Although the construction of consistent O(N) supergravity Lagrangians has been successfully achieved for all N up to $N=8$,[5] the corresponding Lagrangians with gauged internal symmetry have, to date, been written down explicitly only for $N=2$ and 3 (Ref. 6) and $N=4$.[16,17] It is thus an assumption on our part that such Lagrangians exist for $N=5$, 6, 7, and 8. As far as we are aware, there are no theoretical reasons preventing such a construction since the appropriate supersymmetry algebras for $N>4$ are perfectly respectable.[18] (We refrain from going beyond $N=8$ for the usual reason of requiring no spin higher than 2.) The crucial observation, however, is that by restricting our attention to the gravitational part of the on-shell counterterms at the one-loop level, we make the details of the interaction terms in such Lagrangians not relevant: All that is required to determine the coefficients A and B is the pure spin-2 Lagrangian itself together with that part of the remaining Lagrangian quadratic in the lower-spin fields. Once we have calculated the gravitational contribution to ΔS on shell, the remainder is determined by the supersymmetry which guarantees that (with $\kappa^2\Lambda=-6e^2$)

$$\Delta S=-(1/\epsilon)[A\chi+B(e^2/2\pi^2)S], \qquad (7)$$

where S is the classical action. The signal for asymptotic freedom is $B>0$.[9]

Only the kinetic terms are needed to fix the contributions to A from fields of different spin. These have been calculated before.[19] To calculate B we also require knowledge of the mass terms. All particles must be massless for all N if, as we are assuming, supersymmetry is not spontaneously broken. For $N\leqslant 4$ there is an "apparent mass" parameter m for the gravitinos given by $\Lambda = -3m^2$ which we assume to remain the same for $N>4$. Similarly we assign no such parameters to the spin-1 and spin-$\frac{1}{2}$ fields for $N>4$ since they are absent for $N \leqslant 4$. The scalar fields, which first make their appearance at $N=4$, require greater care. The spin-2, spin-0 coupling in the $N=4$ model is known to be minimal with a mass term.[16,17] The special value of the mass permits a Weyl rescaling to a conformal coupling with no mass term.[3] Both versions yield the same B coefficient on the mass shell.[20] We therefore adopt a conformal coupling with no mass term for all $N \geqslant 4$. With the above assumptions, the calculations of Ref. 3 give the contributions to A and B shown in Table I. The combined results for O(N) supergravity then follow from the well-known particle content shown in Table II. The most remarkable feature is clearly the vanishing of the B coefficient for all $N>4$, though the integral value of A for all $N>2$ is not without interest. We now discuss the implications of these results.

Renormalizability.—Apart from the topological χ counterterm, the $N>4$ theories are seen to remain one-loop finite on shell even when the internal symmetry is gauged and $\Lambda \neq 0$.[21] In particular, the one-loop contribution to the renormalization group $\beta(e)$ function vanishes! This is reminiscent of the $N=4$ Yang-Mills multiplet in flat space, whose β function is known to vanish to two-loop order.[22] The vanishing β function in $N>4$ gauged supergravity is no less mysterious than in $N=4$ Yang-Mills and, at the time of writing, is understood only as a "miraculous" cancellation of numerical coefficients. [There have been earlier speculations[23] that $N>4$ theories might show improved ultraviolet behavior, but we do not know their connection, if any, with the concrete calculations presented here.] These cancellations can hardly be accidental, however, and provide something of an *a posteriori* justification for our previous assumptions on $N>4$ theories.

For $N \leqslant 4$, we do not have one-loop finiteness but rather one-loop renormalizability. Moreover, the negative value of B indicates that these theories are not asymptotically free, inasmuch as asymptotic freedom is meaningful for theories which may not be renormalizable at higher loops. One will find two-loop renormalizability for all N when $\Lambda \neq 0$ for the same reason one finds two-loop finiteness when $\Lambda = 0$,[24] and it would be interesting to know the β function. Three loops and beyond is still a mystery.[25]

Topology.—Another remarkable feature peculiar to $N>4$ models is that $\gamma = A\chi + B\delta =$ integer (since A is an integer, B is zero, and χ takes on integer values). This may be indicative of a new "super index theorem"[26] for $N>4$. Let us recall the significance of γ.[27,26] At the one-loop level γ counts the total number of eigenmodes (boson minus fermion) of the differential operators whose determinants govern the one-loop functional integral. (It is closely related to the anomalous trace of the energy-momentum tensor.) The number of zero-eigenvalue modes will be finite and given by an integer; the number of nonzero modes is formally infinite. After regularization (e.g., by the ζ-function method), this number is rendered finite but not necessarily an integer. In certain circumstances, however, there may be a mutual cancellation of the nonzero modes between the bosons and fermions, in which case $\gamma =$ integer. Such a cancellation does indeed take place in $\Lambda = 0$ supergravity[27] when $R_{\mu\nu\rho\sigma} = \pm {}^*R_{\mu\nu\rho\sigma}$ (which implies $R_{\mu\nu} = 0$). If, in addition, the space is compact with spin structure (i.e., fermions can be globally defined) then $\chi =$ integer $\times 24$. Consistent with this is the result in Table II that $A =$ integer/24 for all N. We do not know whether any similar

TABLE I. Contributions to the coefficients A and B from fields of spin S.

S	$360A$	$60B$
0	4	-1
1/2	7	-3
1	-52	-12
3/2	-233	137
2	848	-522

TABLE II. The particle spectra in O(N) supergravity and the corresponding values of A and B.

N \ S	2	3/2	1	1/2	0	A	B
1	1	1				41/24	-77/12
2	1	2	1			11/12	-13/3
3	1	3	3	1		0	-5/2
4	1	4	6	4	2	-1	-1
5	1	5	10	11	10	-2	0
6	1	6	16	26	30	-3	0
7	1	8	28	56	70	-5	0
8	1	8	28	56	70	-5	0

mechanism can take place when $\Lambda \neq 0$ and χ is not so restricted, but our results indicate that $N > 4$ models are the most likely candidates.

Finally, we note that the signs of A and B in simple and extended supergravity reinforce the conclusions concerning "space-time foam" reached in the context of pure gravity.[1] If these one-loop results are taken seriously, the sign of γ would seem to imply that space-time becomes "foamier and foamier" the shorter the length scale, in contrast to the picture of "one unit of topology per Planck volume" expected if γ were positive definite.[7]

This work was supported in part by the National Science Foundation under Grant No. PHY-77-27084.

Note added.—Recent work on antisymmetric tensors has shown that the A coefficient depends not only on the spin but also on the choice of field representation.[28] Thus $A[\varphi_{\mu\nu}] = A[\varphi] + 1$ and $A[\varphi_{\mu\nu\rho}] = -2$. Moreover, the representation content of the $N = 8$ theory obtained by dimensional reduction from eleven dimensions replaces the 70 φ fields by 63 φ plus 7 $\varphi_{\mu\nu}$ plus $\varphi_{\mu\nu\rho}$. Hence $A = -5 + 7 - 2 = 0$.[29]

[(a)]Permanent address.

[1]S. M. Christensen and M. J. Duff, Nucl. Phys. B (to be published).

[2]G. W. Gibbons and J. M. Perry, Nucl. Phys. B146, 90 (1978). This paper contains numerical errors corrected in Ref. 1.

[3]S. M. Christensen, M. J. Duff, G. W. Gibbons, and M. Roček, to be published.

[4]D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, Phys. Rev. D 13, 3214 (1976); S. Deser and B. Zumino, Phys. Lett. 62B, 335 (1976).

[5]See, for example, J. Scherk, in *Proceedings of the Summer Institute on Recent Developments on Gravitation, Cargèse, France, 1978,* edited by M. Levy *et al.* (Plenum, New York, 1979).

[6]D. Z. Freedman and A. Das, Nucl. Phys. B120, 221 (1977); E. S. Fradkin and M. Vasiliev, to be published.

[7]S. W. Hawking, Nucl. Phys. B144, 349 (1978). See also P. K. Townsend, Phys. Rev. D 15, 2795 (1977).

[8]B. S. DeWitt, in *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[9]G. 't Hooft, Nucl. Phys. B62, 444 (1973).

[10]G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré 20, 69 (1974).

[11]P van Nieuwenhuizen, in *Proceedings of the Summer Institute on Recent Developments in Gravitation, Cargèse, France, 1978,* edited by M. Levy *et al.* (Plenum, New York, 1979). Previous discussions of supergravity with a cosmological constant may be found in S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. 38, 739, 1376(E) (1977); S. Deser and B. Zumino, Phys. Rev. Lett. 38, 1433 (1977); P. K. Townsend, Phys. Rev. D 15, 2802 (1977).

[12]We do not anticipate any problems in this respect of the kind discussed by B. deWit and M. T. Grisaru, Phys. Rev. D 20, 2082 (1979), since our calculations are insensitive to the elimination of the auxiliary fields.

[13]P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D 19, 3592 (1979).

[14]M. Grisaru, P. van Nieuwenhuizen, and J. A. M. Vermaseren, Phys. Rev. Lett. 37, 1662 (1976).

[15]Note that even in nonsupersymmetric theories the β function will change when one allows for gravity and a nonvanishing Λ. Thus (modulo renormalizability problems) a theory which is asymptotically free in flat space might be converted into one which is not and vice versa.

[16]A. Das, M. Fischler, and M. Roček, Phys. Rev. D 16, 3427 (1977).

[17]D. Z. Freedman and J. H. Schwarz, Nucl. Phys. B137, 333 (1978).

[18]W. Nahm, Nucl. Phys. B135, 149 (1978).

[19]See S. M. Christensen and M. J. Duff, Phys. Lett. 76B, 571 (1978), and references therein.

[20]In neither version does the scalar potential contain terms linear in φ. In this respect, and also in that it truncates consistently to the O(3) and O(2) models, the O(4) model differs from the alternative chiral SU(2) $\otimes$SU(2) gauged supergravity of Ref. 18, which will not be discussed here. Both theories apparently suffer from a scalar potential $V(\varphi)$ which is not bounded below. Since $V(\varphi)$ is intimately connected with Λ, however, it may be that the criterion for stability is different when $\Lambda \neq 0$.

[21]Note that, *a priori*, our results have nothing to do with those of E. Cremmer, J. Scherk, and J. Schwarz, Phys. Lett. 84B, 83 (1979), who found that the Λ induced by one-loop quantum corrections was finite in the *spontaneously broken* version of *ungauged* extended supergravity which has *zero* Λ at the classical level.

[22]D. R. T. Jones, Phys. Lett. 72B, 199 (1977); H. Pendleton and E. Poggio, Phys. Lett. 72B, 200 (1977). See also M. Grisaru, W. Siegel, and M. Roček, Nucl. Phys. B159 (1979).

[23]B. deWit and S. Ferrara, Phys. Lett. 81B, 317 (1979).

[24]M. Grisaru, Phys. Lett. 66B, 75 (1977).

[25]S. Deser, J. Kay, and K. Stelle, Phys. Rev. Lett. 38, 527 (1977).

[26]S. M. Christensen and M. J. Duff, Nucl. Phys. B154, 301 (1979).

[27]S. W. Hawking and C. N. Pope, Nucl. Phys. B146, 381 (1978).

[28]M. J. Duff and P. van Nieuwenhuizen, to be published.

[29]W. Siegel, private communication.