Scaling Behavior of Chaotic Flows

B. A. Huberman
Xerox Palo Alto Research Center, Palo Alto, California 94304

and

J. Rudnick
Physics Department, University of California, Santa Cruz, California 95064
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It is shown that in the turbulent regime of systems with period-doubling subharmonic bifurcations, the maximum Lyapunov characteristic exponent behaves like $\bar{\lambda}=\bar{\lambda}_0(r-r_c)^t$, with t a universal exponent which is calculated to be $t = 0.449\,806\,9\ldots$. This result is in agreement with the available data on $\bar{\lambda}$ for a number of dynamical systems.

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There exist a large number of physical systems for which the nonlinear equations describing their dynamics display transitions into a chaotic regime in the absence of external noise sources. This regime, which is characterized by broadband noise in the power spectral densities, has been extensively studied in simple fluids, plasmas, chemical reactions, and various mathematical models.[1] In the case of dissipative systems, a pervasive pathway to turbulent behavior appears to be made of a cascade of period-doubling subharmonic bifurcations into a strange attractor in phase space. This has been observed in some experiments on the onset of fluid turbulence,[2-4] studies of driven anharmonic oscillators,[5] nonlinear saturation of unstable plasma modes,[6] and several other mathematical models.[7-9] This cascading behavior in the periodic regime has been shown to display universal features, independent of the detailed nature of the governing equations.[10] More recently, it has been established[11-13] that beyond the onset of chaos another set of fiburcations takes place whereby 2^n bands of the attractor successively merge in a mirror sequence of the cascading bifurcations found in the periodic regime.

A hallmark of ergodic and mixing behavior for nonlinear dynamical systems is their sensitive dependence on initial conditions. Two trajectories in phase space that initially differ by a small amount will separate exponentially in time, with the divergence rate measured by a positive value of the maximum Lyapunov characteristic exponent, $\bar{\lambda}$, associated with the flow.[14,15] For systems that display period-doubling subharmonic bifurcations, the emergence of a positive value for the envelope of $\bar{\lambda}$ as the control parameter r exceeds the onset value r_c, takes place in a steep and continuous fashion, a behavior reminiscent of critical-point phenomena in phase transitions. As illustrated in Fig. 1, which has been computed for a one-dimensional map, as $r\to r_c^+$ the envelope of $\bar{\lambda}$ seems to approach its zero value with power-law behavior, the sharp dips corresponding to stable orbits in the chaotic regime.

In this paper we show that the power-law behavior for the envelope of $\bar{\lambda}$ suggested by Fig. 1 is indeed universal for dynamical systems exhibiting period-doubling subharmonic bifurcations; the characteristic exponent behaves as

$$\bar{\lambda}=\bar{\lambda}_0(r-r_c)^t \tag{1}$$

with $\bar{\lambda}_0$ a constant of order unity and t an expo-

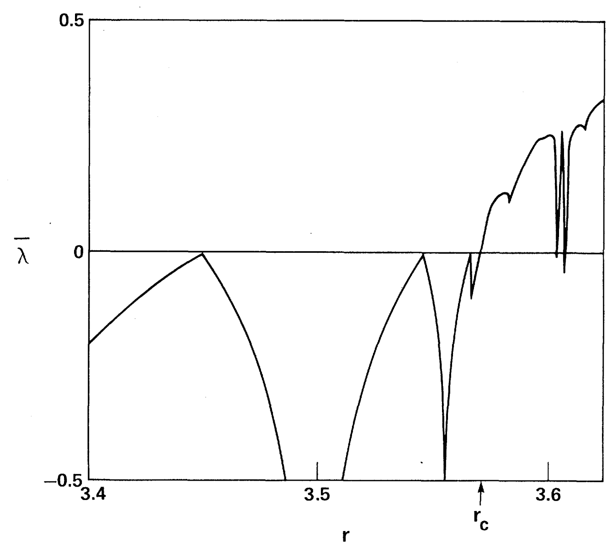

FIG. 1. The Lyapunov characteristic exponent for the one-dimensional map $x_{k+1}=rx_k(1-x_k)$, with $0\leq x_k\leq 1$ and $r_c=3.57$. The sharp dips in the region $r>r_c$ correspond to periodic orbits. (See Ref. 18.)

nent which is given by

$$t=\frac{\ln 2}{\ln\delta}=0.449\,806\,9\ldots, \tag{2}$$

where δ is a constant which has been determined to be given by $\delta=4.669\,261\,6\ldots$ (Ref. 10). This result is in agreement with all present computations and measurements of $\bar{\lambda}$ (albeit not extremely accurate) available for these systems.

Consider a one-dimensional map,[16] defined by $x_{k+1}=f(x_k, r)$ with $f(x,r)$ a continuous, single-hump function with a parabolic maximum in the interval $0\le x\le 1$, and r a variable that controls its steepness. If $f^{(n)}(x,r)$ denotes the nth iterate of the map, the Lyapunov characteristic exponent, $\bar{\lambda}(r)$, can be written as

$$\bar{\lambda}(r)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}2^{-n}\ln\left|\frac{df(x_k,r)^{(2^n)}}{dx_k}\right|. \tag{3}$$

If we assume that there exists a probability distribution, $p_r(x)$, that is invariant under the operations of the map [i.e., $p_r(x_i)=\sum_k p_r(x_{i-1}{}^k)dx_{i-1}{}^k/dx_i$, where the sum is over the x_{i-1}'s that are mapped into x_i], we can express Eq. (3) as

$$\bar{\lambda}(r)=2^{-n}\int p_r(x)\ln\left|\frac{df(x,r)^{(2^n)}}{dx}\right|dx. \tag{4}$$

In the chaotic regime (i.e., $r>r_c$) the existence of the reverse set of bifurcations that we described above implies that up to the $(n+1)$ bifurcation, $f^{(2^n)}(x_i,r)$ maps points within a given band of the attractor into the same band. Moreover, since the distance between bands scales like α^{-n} with α a universal constant,[10] the probability distribution $p(x,r)$ will consist of a set of 2^n narrow strips of width α^{-n} and height h, separated by regions in which it is identically zero. This in turn implies that Eq. (4) can be broken into 2^n integrals over each band evaluated with $f^{(2^n)}(x,r)$. If in the spirit of the Feigenbaum scaling study of the periodic regime[10] we assume that $f^{2^n}(x,r)-x=\alpha^{-n}\varphi[\alpha^n x,\delta^n(r-r_c)]$ with φ a universal function of x, we can write Eq. (4) as

$$\bar{\lambda}(r)=\alpha^{-n}\int_{\substack{\text{single}\\ \text{band}}}p(y/\alpha^n)\times\ln|\varphi'[y,\delta^n(r-r_c)]+1|\,dy, \tag{5}$$

where $y=\alpha^n x$ and $\delta^n(r-r_c)$ is of order unity. In order to evaluate this integral, first note that the normalization condition for the probability function written as

$$2^n\int_{\substack{\text{single}\\ \text{band}}}p(x)\,dx\simeq 2^n\alpha^{-n}h=1 \tag{6}$$

enables one to write $p(y/\alpha^n)$ as $hR(y)$, where $R(y)$ is a uniform function of y over the width of the band, and $h=\alpha^n 2^{-n}$. Equation (5) then becomes

$$\bar{\lambda}(r)=2^{-n}\bar{\lambda}_0'[\delta^n(r-r_c)] \tag{7}$$

where the functional dependence of $\bar{\lambda}_0'$ on $\delta^n(r-r_c)$ describes the structure beneath the envelope. Since we are not dealing with that structure we replace it by a constant $\bar{\lambda}_0''$. We therefore see that $\bar{\lambda}$ increases as the number of bands within the strange attractor merge pairwise into a single one.

In order to obtain a scaling relation that involves the control parameter r, we note that in the highly bifurcated regime the value of r for which the nth bifurcation takes places behaves like[10]

$$r-r_c=c\delta^{-n} \tag{8}$$

with c a constant, so that $n=c'+\ln(r-r_c)/(-\ln\delta)$. Use of this equality in Eq. (7) results in

$$\bar{\lambda}(r)=\bar{\lambda}_0 2\ln(r-r_c)/\ln\delta, \tag{9}$$

where $\bar{\lambda}_0=2^{-c'}\bar{\lambda}_0''$ or, equivalently

$$\bar{\lambda}(r)=\bar{\lambda}_0(r-r_c)^t \tag{10}$$

with the universal exponent t, given by $t=\ln 2/\ln\delta=0.449\,806\,9\ldots$. Since points in phase space separated by an initial distance d will separate after m iterations of the map like $Me^{\bar{\lambda}m}$ with M a constant, Eq. (10) expresses the fact that the rate of divergence will grow like a power law as one enters the turbulent regime.

There exists at the present time a number of calculations[8,11,17,18] and measurements of $\bar{\lambda}$ as a function of r for many different systems displaying cascades of period-doubling bifurcations into a chaotic state. To within the accuracy with which we can compare them with the predictions of our scaling theory, they are all in good agreement with Eq. (10). It is clear, however, that more accurate calculations and measurements will have to be made in order to have a precise test of the theory. Furthermore, since the effect of external noise is to produce a bifurcation gap in the sequence of available states[13] the scaling behavior of $\bar{\lambda}$ can only be checked in the limit of small fluctuations or truncation errors. This appears quite feasible.

In concluding, we point out that the theory we have presented applies only to a region near the onset value r_c for which the mirror sequence of period-doubling bifurcation takes place. Although in some dynamical systems[11,13] this sequence exhausts most of the turbulent regime, it is well known that there are other nonlinear problems for which there exist beyond the single-band attractor a rich structure of orbits and bands, and where our arguments may no longer apply. In that region the observed growth of the envelope of $\bar{\lambda}$ as a function of r, although it still reflects the effects of larger bandwidths, will require a different theoretical approach from the one presented here. Nevertheless it is rewarding to have a measure of chaos whose universal behavior near onset is exactly calculable.

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