similar microwave magnetic-resonance transitions in $(^3He^{++}\mu^- e^-)^0$, in which the value of $\Delta\nu$ will be determined in part by the magnetic moment of the ^{3}He nucleus.[8]

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First-Passage–Time Distributions under the Influence of Quantum Fluctuations in a Laser

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The distribution of first-passage times is calculated for a homogeneously broadened two-mode laser, that is characterized by a bistable potential. As a consequence of quantum fluctuations, such a system tends to switch spontaneously between the two metastable states. The results of the calculation are compared with first-passage–time measurements of a two-mode dye laser.

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The behavior of a system subject to fluctuations under the influence of a double potential well, that provides two metastable states, is a fundamental problem in physics. Examples of such bistable systems can be found in fluid dynamics, thermodynamics, and quantum optics, and they are frequently associated with first-order phase transitions.[1] The problem of determining the rate at which the system switches between bistable states is the classic first-passage–time problem, which has been treated in numerous papers.[2,3] However, there appear to be few examples in which the probability distribution of the first-passage times can be easily calculated and compared with measurements. We wish to describe a simple physical system, a laser oscillating in two modes, for which both the calculation and the associated experiment have been carried out. The measured average first-passage times, covering a range of nearly five orders of magnitude, and the measured probability distributions well above threshold, provide the first confirmation of the theory.

We consider a laser oscillating simultaneously in two modes with dimensionless complex amplitudes E_1, E_2. Because of the spontaneous emission fluctuations, E_1, E_2 behave as random processes governed by Langevin equations, whose joint probability density $p(E_1, E_2, t)$ obeys a four-dimensional Fokker-Planck equation. When the equation is written in scaled form[4,5] it contains three parameters, the pump parameters a_1, a_2 for the two laser modes, and the mode coupling constant ξ, which can exceed unity when the atomic or molecular system is homogeneously broadened, and becomes 2 for a ring laser of this type. Although the general solution of the four-dimensional Fokker-Planck equation has so far only been found when $\xi=1$,[5] the steady-state solution is easily obtained. If $I_1 \equiv |E_1|^2$ and $I_2 \equiv |E_2|^2$ are the intensities of the two laser modes, the probability density $\mathcal{P}(I_1)$ of I_1 in the steady state is given by[6,7]

$$\mathcal{P}(I_1) = \pi^{1/2} Q^{-1} \exp[\tfrac{1}{4}(a - \tfrac{1}{2}\Delta a)^2] \exp[-V(I_1)], \quad (1)$$

where Q is a normalizing constant and $V(I)$ is the potential

$$V(I, \alpha, \Delta a, \xi) = \tfrac{1}{4}(\xi^2 - 1)I^2 + \tfrac{1}{2}I[a(\xi - 1) - \tfrac{1}{2}\Delta a(\xi + 1)] - \ln[1 - \mathrm{erf}(\tfrac{1}{2}(\xi I - a + \tfrac{1}{2}\Delta a))]. \quad (2)$$

We have used the abbreviations $\frac{1}{2}(a_1 + a_2) \equiv a$, $a_1 - a_2 \equiv \Delta a$. If $\xi > 1$, the potential $V(I)$ exhibits two dips corresponding to the two metastable states, one at $I=0$ and one at $I \approx a + \frac{1}{2}\Delta a$, with a maximum in between at $I = I_A$ with

$$I_A \approx a/(\xi + 1) - \tfrac{1}{2}\Delta a/(\xi - 1). \quad (3)$$

When a sufficiently large fluctuation occurs the system switches rapidly from one of the two bistable states to the other. Moreover, the intensities of the two laser modes are strongly anticorrelated when $\xi = 2$.[7]

As the total range of I is divided into two by the value $I = I_A$, we may deem a one-dimensional first passage from some initial value I_0 to occur when I first crosses the boundary $I = I_A$, in either direction. The justification for treating the first-passage problem as one dimensional, to a first approximation, lies in the tendency of the representative point in the four-dimensional phase space to follow a trajectory close to the most probable one. The motion is therefore almost one dimensional. The validity of the approximation should improve steadily above threshold. The probability density $P(T, I_0)$ of the first-passage time $T(I_0)$ from I_0 can be shown to obey the adjoint Fokker-Planck equation,[2,3] and in one dimension the rth moment of $T(I_0)$ satisfies[2,3]

$$D d^2\langle T^r\rangle/dI_0^2 + B d\langle T^r\rangle/dI_0 = -r\langle T^{r-1}\rangle, \quad r = 1, 2, \ldots . \quad (4)$$

$D(I)$ and B are diffusion and drift coefficients, respectively, of the one-dimensional Fokker-Planck equation for $\mathcal{P}(I, t)$ and $D(I) = 4I$ in our notation.[6] When $r = 1$ the equation can be integrated directly, and we find,[6-8] when $I_0 < I_A$,

$$\langle T(I_0)\rangle = \int_{I_0}^{I_A} dI' \frac{1}{\mathcal{P}(I')D(I')} \int_0^{I'} dI'' \mathcal{P}(I''), \quad (5)$$

and, when $I_0 > I_A$,

$$\langle T(I_0)\rangle = \int_{I_A}^{I_0} dI' \frac{1}{\mathcal{P}(I')D(I')} \int_{I'}^{\infty} dI'' \mathcal{P}(I''). \quad (6)$$

These mean values can now be used to generate the higher-order moments recursively from Eq. (4). If the initial state is the "off" state $I = 0$, we obtain, for $r = 1, 2, \ldots$,

$$\langle T_{\mathrm{off}}{}^r\rangle = r! \int_0^{I_A} dI_{2r} \frac{1}{\mathcal{P}(I_{2r})D(I_{2r})} \int_0^{I_{2r}} dI_{2r-1} \mathcal{P}(I_{2r-1}) \times \int_{I_{2r-1}}^{I_A} dI_{2r-2} \cdots \int_0^{I_4} dI_3 \mathcal{P}(I_3) \int_{I_3}^{I_A} dI_2 \frac{1}{\mathcal{P}(I_2)D(I_2)} \int_0^{I_2} dI_1 \mathcal{P}(I_1), \quad (7)$$

whereas from the initial "on" state $I_0 = a + \frac{1}{2}\Delta a$, we have

$$\langle T_{\rm on}{}^r \rangle = r\,! \int_{I_A}^{a+\Delta a/2} dI_{2r} \frac{1}{\mathcal{P}(I_{2r})D(I_{2r})} \int_{I_{2r}}^{\infty} dI_{2r-1}\mathcal{P}(I_{2r-1}) \int_{I_A}^{I_{2r-1}} dI_{2r-2} \cdots \int_{I_4}^{\infty} dI_3\, \mathcal{P}(I_3) \times \int_{I_A}^{I_3} dI_2 \frac{1}{\mathcal{P}(I_2)D(I_2)} \int_{I_2}^{\infty} dI_1 \mathcal{P}(I_1) . \quad (8)$$

The evaluation of the integrals is greatly simplified by the fact that, within the range $I < I_A$, $\mathcal{P}(I)$ is very small except near $I = 0$, and falls as I approaches I_A, and in the range $I > I_A$, $\mathcal{P}(I)$ is very small except near $I = a + \frac{1}{2}\Delta a$, and again falls towards $I = I_A$. Therefore the I' integrand in Eq. (5), for example, makes its greatest contribution near $I' = I_A$, in which case the I'' integral is almost independent of I', and may be replaced by $P_{\rm low} = \int_0^{I_A} dI''\, \mathcal{P}(I'')$, the probability that the light intensity lies below I_A, which has already been calculated.[7] Equations (5) and (6) therefore lead to

$$\begin{aligned} \langle T_{\rm off}\rangle &= P_{\rm low} \int_0^{I_A} dI' \,[\mathcal{P}(I')D(I')]^{-1}, \\ \langle T_{\rm on}\rangle &= P_{\rm high} \int_{I_A}^{a+\Delta a/2} dI' \,[\mathcal{P}(I')D(I')]^{-1}, \end{aligned} \quad (9)$$

with $P_{\rm low} + P_{\rm high} = 1$. In the neighborhood of $I = I_A$, the reciprocal probability $1/\mathcal{P}(I')$ behaves almost as a Gaussian distribution centered at I_A, and if we approximate $1/D(I')$ by its value $1/4I_A$ in the neighborhood of the Gaussian peak, we can immediately evaluate the integrals in Eq. (9), which have almost identical values. With the help of the expressions for $P_{\rm low}$ and $P_{\rm high}$, we then arrive at

$$\langle T_{\rm off}\rangle \approx \pi^{1/2} \exp[\tfrac{1}{4}(\xi^2 - 1)I_A{}^2]/[2I_A{}^2(\xi^2-1)^{3/2}], \quad (10)$$

and a closely related expression for $\langle T_{\rm on}\rangle$, which should hold with increasing accuracy as the pump parameter a increases. The same arguments that led us to separate the integrals in Eqs. (5) and (6) to a good approximation, can also be used to separate the multiple integrals in Eqs. (7) and (8), because the integrand in each double integral is peaked near $I = I_A$. We readily find

$$\langle T_i{}^r\rangle = r\,!\,\langle T_i\rangle^r, \quad i = \text{off, on}, \quad (11)$$

from which it follows immediately that

$$P(T_i) = \langle T_i\rangle^{-1} \exp(-T_i/\langle T_i\rangle), \quad i = \text{off, on}, \quad (12)$$

so that both dwell time distributions are exponential in the limit of large a.

We have tested these predictions by measurements on a dye laser oscillating in two longitudinal modes, each of which can be separately studied with the help of a tuned Fabry-Perot interferometer inserted in the output beam, as shown in Fig. 1. The low-gain photodetector gives an output signal proportional to the intensity I of one laser mode, which is passed on to a limiter after some integration. The limiter output consists of a succession of rectangular pulses with variable duration equal to the on time $T_{\rm on}$ of the laser mode. The reading of the integrating meter provides a measure of $\langle T_{\rm on}\rangle/(\langle T_{\rm on}\rangle + \langle T_{\rm off}\rangle)$, whereas the average rate of arrival of pulses yields $1/(\langle T_{\rm on}\rangle + \langle T_{\rm off}\rangle)$, from which both $\langle T_{\rm on}\rangle$ and $\langle T_{\rm off}\rangle$ can be found. Figure 2 shows the results of such an experimental determination of $\langle T_{\rm off}\rangle$, over a range of laser pump parameters a from about 2 to 14. a was determined from the measured light intensity $\langle I\rangle$ and its known dependence on a.[5,6] The laser was operated with the two mode intensities almost equal on the average, so that $\langle T_{\rm on}\rangle \sim \langle T_{\rm off}\rangle$. Also shown in Fig. 2 is the theoretical form of $\langle T_{\rm off}\rangle$ given by Eq. (9) with the coupling constant $\xi = 2$, and with the vertical scale adjusted for best fit between theory and experiment. It will be seen that there is reasonable agreement over dwell times ranging from a few milliseconds to almost a minute. The longest dwell times $\langle T_{\rm off}\rangle$ are, of course, particularly sensitive to external perturbations. $\xi = 1.8$ gives almost as good a fit with the data, but lower values of ξ do

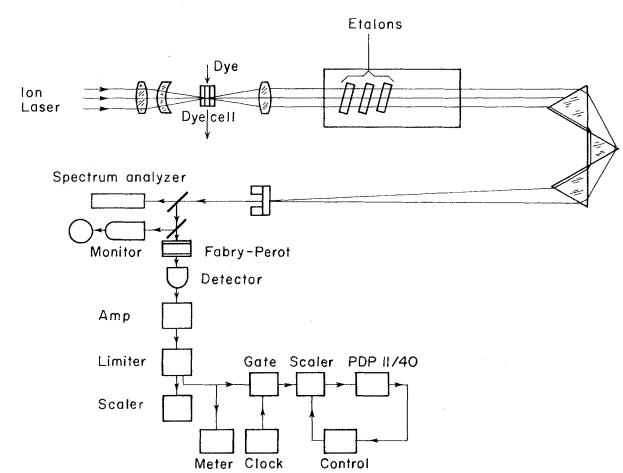

FIG. 1. Outline of the apparatus.

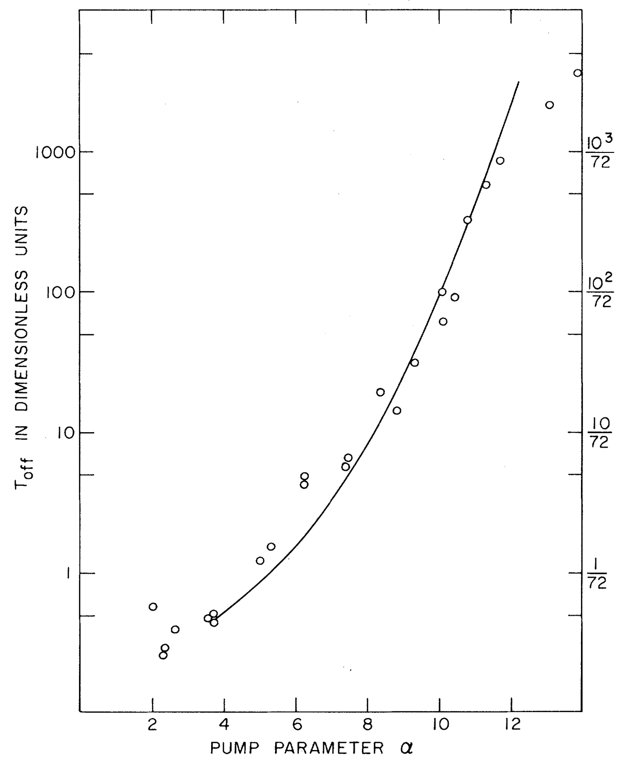

FIG. 2. Average first-passage time $\langle T_{\text{off}} \rangle$ as a function of pump parameter a with $\Delta a \ll 1$. The circles are experimental points, and the right-hand scale gives measured values of T_{off} in seconds. The full curve is theoretical and given by Eq. (9).

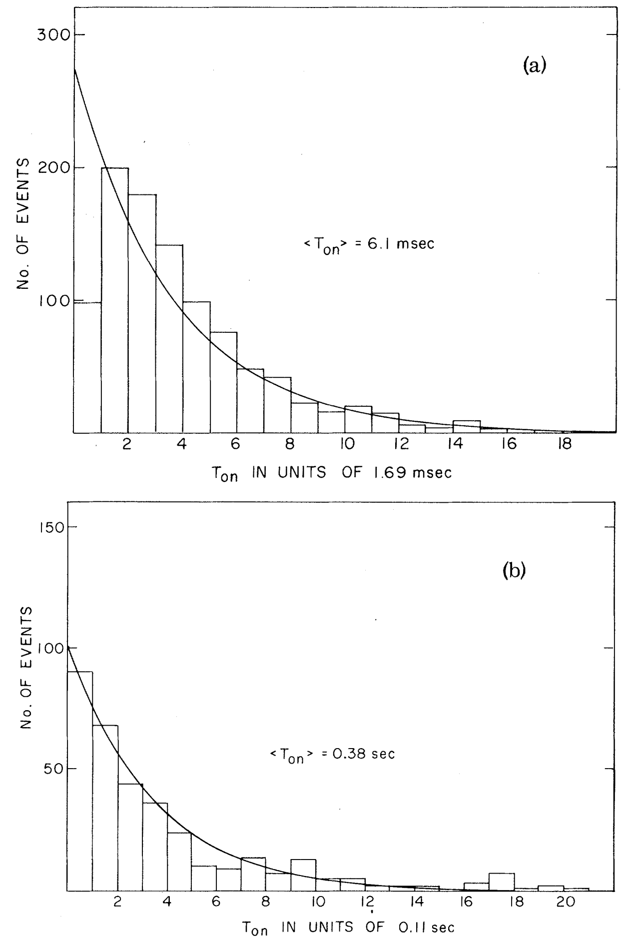

FIG. 3. Histograms of the measured probability distributions $P(T_{\text{on}})$ for pump parameters (a) $a \approx 3.5$; (b) $a \approx 9$, with the theoretical values for the same mean given by Eq. (12) superimposed as full curves.

not. Although ξ can, in principle, be determined from the laser geometry,[10] there are large uncertainties in practice.

Next the probability distribution $P(T_{\text{on}})$ of the dwell time T_{on} was determined by measuring the distribution of the number of clock pulses from a clock of adjustable frequency that occurred within the time intervals T_{on}, when the start of the interval coincided with a clock pulse. The counting system is illustrated in Fig. 1, and it is similar to the system described in Ref. 9. If the clock frequency is f, n clock pulses are counted when the time interval T_{on} lies in the range $n/f \leq T_{\text{on}} < (n+1)/f$, $n = 0, 1, 2, \ldots$. After many repetitions, the number of events with n clock pulses becomes a measure of the probability $\int_{n/f}^{(n+1)/f} P(T)\,dT$. Figure 3 gives the results of measurements at two different pump parameters, for which $\langle T_{\text{on}} \rangle$ ranged over almost two orders of magnitude; superimposed as full curves are the distributions given by Eq. (12) with the same means.

The results indicate that $P(T_{\text{on}})$ tends to become exponential as the pump parameter a and the average dwell time $\langle T_{\text{on}} \rangle$ increase, as predicted by Eq. (12). Departures from the exponential form are more noticeable closer to threshold, where the asymptotic approximation to Eqs. (7) and (8) is less valid, and at the shortest dwell times, where the approximation of treating the first-passage problem as one dimensional is most suspect. The measurements confirm that the metabistability of a homogeneously broadened two-mode laser may be regarded as a first-passage phenomenon. Both the calculation and the experimental technique should be applicable to other bistable systems.

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Efficiency Factors in Mie Scattering

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Asymptotic approximations to the Mie efficiency factors for extinction, absorption, and radiation pressure, derived from complex–angular-momentum theory and averaged over $\Delta\beta \sim \pi$ (β = size parameter), are given and compared with the exact results. For complex refractive indices $N = n + i\kappa$ with $1.1 \leq n \leq 2.5$ and $0 \leq \kappa \leq 1$, the relative errors decrease from $\sim(1-10)\%$ to $\sim(10^{-2}-10^{-3})\%$ between $\beta = 10$ and $\beta = 1000$, and computing time is reduced by a factor of order β, so that the Mie formulae can advantageously be replaced by the asymptotic ones in most applications.

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The Mie efficiency factors[1] for extinction (Q_{ext}), absorption (Q_{abs}), and radiation pressure (Q_{pr}) are just the corresponding cross sections divided by the projected area πa^2 of the scattering sphere. These quantities are important in many applications. Typical size parameters $\beta = ka$ (k = wave number, a = droplet radius) range from $\ll 1$ up to $\sim 10^4$, with complex refractive indices $N = n + i\kappa$, $1.1 \lesssim n \lesssim 1.9$, $10^{-9} \lesssim \kappa \lesssim 1$. The efficiencies vary extremely rapidly[2] with β, n and κ; but in most applications one is only interested in means $\langle Q \rangle$ over some range $\Delta\beta$, not in this high-frequency "ripple."

Evaluation of the exact Mie expressions[1] requires summing $\sim\beta$ partial waves. Upon integration across size or wavelength with a step fine enough to resolve the ripple ($\Delta\beta \leq 0.01-0.1$), one is faced with exorbitant computation times. Approximations[1] based on geometrical optics and classical diffraction theory do not have the required accuracy until β exceeds several thousand (cf. below). Clearly, better approximations, devoid of ripple, are needed.

The complex-angular-momentum theory of Mie scattering[3] can furnish such approximations. By a simple extension of previously developed tech-