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Hydrodynamic Theory of Mutual Friction in He II

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Theoretical values of the mutual-friction parameters B and B' are derived from a purely hydrodynamic model, in close agreement with experimental data in the temperature range from 1.7 to 2.1 K.

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In this paper we wish to consider the small two-dimensional motion $\vec{r}_L(t)$ of a vortex line subject to the normal and superfluid velocity fields $\vec{v}_{n1}$ and $\vec{v}_{s1}$ of a second-sound wave. Its velocity $\vec{v}_L$ will be given by[1]

$$d\vec{r}_L/dt = \vec{v}_L = \vec{v}_{s1} - \frac{B\rho_n}{2\rho}\,\vec{\nu}\times(\vec{v}_{s1}-\vec{v}_{n1}) - \frac{B'\rho_n}{2\rho}(\vec{v}_{s1}-\vec{v}_{n1}), \quad (1)$$

where ρ is the helium density, ρ_n the normal fluid density, and $\vec{\nu}$ is a unit vector along the vortex line. Equation (1) may be regarded as the very definition of the two mutual-friction coefficients B and B', first introduced by Hall and Vinen. In their original paper,[2] Hall and Vinen (H-V) worked out a detailed microscopic model for B and B'. This model combines a kinetic treatment of the rotons-vortex collisions with purely hydrodynamic arguments (Magnus effect, dragging of the normal fluid), and leads to intricate relationships giving B and B' as a function of the effective collision diameters $\sigma_\parallel$ and $\sigma_\perp$ describing the scattering of rotons by vortices.[3] In a range of temperature around 1.9 K (i.e., 1.9 $\pm$ 0.2 K), where it is known from experiment[4] that $B'/B \lesssim 0.1$, the H-V expression of B can be simplified, and then B written, with accuracy better than 1%, as a complex quantity of the form

$$B = B_1 + iB_2 = \left[A - \frac{\rho_n\rho_s K}{16\pi\eta\rho}\left(\ln\omega - \frac{i\pi}{2}\right)\right]^{-1}. \quad (2)$$

Here $\rho_s = \rho - \rho_n$, η is the coefficient of normal viscosity, K the quantum of circulation, and $\omega/2\pi$ is the second-sound frequency [according to notations of Ref. 3, $B = (2\rho/\rho_n\rho_s K)X^{-1}$, if quantities of the order of $(B'/B)^2 = (Y/X)^2 \lesssim 10^{-2}$ are neglected]. The main term A in Eq. (2)—$A \sim 1$—is a real and frequency-independent quantity, appearing in explicit form as a rather complicated expression involving $\sigma_\parallel$ and $\sigma_\perp$, the roton-roton mean free path L, and also the normal viscosity η.[3] The other two explicit terms ($\sim 5\times10^{-2}$ typically) represent small corrections usually ignored when analyzing experimental results. In particular, the sound frequency ω is not systematically specified in collected data B vs T. However, it has been pointed out by Mehl *et al.*[5] that the slight frequency dependence of the attenuation of second sound, observed in a rotating cavity, and the apparent variation due to rotation of the second-sound velocity are well accounted for by the $\ln\omega$ term and the imaginary term in (2), respectively. Improving the accuracy of measurements still further and enlarging the worked frequency range, we ourselves have studied B_1 and B_2 vs ω at 1.9 K.[6] Our results corroborate those of Mehl *et al.*, and provide strong support for the correctness of the last two terms in the H-V formula (2). In contrast, none of the theoretical values of $\sigma_\parallel$ and $\sigma_\perp$ derived in literature can account for the principal term A, i.e., for the absolute value of B, as shown in the paper of Hillel *et al.*[3] (see, for example, their Fig. 1).

Both terms $\ln\omega$ and $i\pi/2$ in (2) originate in the same quantity noted $1/E$ in Refs. 2 and 3, which itself arises from the simple model of a solid wire dragging an ordinary viscous fluid. Curiously enough this term $1/E$, originally introduced with due caution by the authors, turns out to be the only one agreeing quantitatively with experiment. Therefore we were led to formulate a purely hydrodynamic theory of mutual friction starting from the Landau-Khalatnikov two-fluid equations. This was also suggested by Putterman in his book,[7] but on the basis of too restrictive assumptions about the normal fluid flow.

Thus, let us suppose we are justified in applying Landau equations even in the vortex-core region, fully aware though that such a procedure certainly fails at low temperature, since the roton-roton mean free path L becomes large compared to the core radius. We shall pay particular attention to the consequence, in vortex dynamics, of the laws of mass and momentum conservation. Assuming the motion of an incompressible fluid ($\rho=\text{const}$, $\nabla\rho_s=-\nabla\rho_n$), which is consistent with the second-sound wave field, these two laws may be written as

$$\nabla\cdot(\rho_n\vec{v}_n+\rho_s\vec{v}_s)=0\,, \tag{3a}$$

$$\rho_n\left[\frac{\partial\vec{v}_n}{\partial t}+(\vec{v}_n\cdot\nabla)\vec{v}_n\right]+\rho_s\left[\frac{\partial\vec{v}_s}{\partial t}+(\vec{v}_s\cdot\nabla)\vec{v}_s\right]+(\vec{v}_s-\vec{v}_n)\left[\frac{\partial\rho_s}{\partial t}+\nabla\cdot(\rho_s\vec{v}_s)\right]=-\nabla p+\nabla\cdot\vec{\sigma}\,, \tag{4a}$$

where p is the pressure and $\vec{\sigma}$ the viscous stress tensor.[8] We shall adopt the simple model of a superfluid density ρ_s varying from $\rho_s=\rho_{s0}=\text{const}$ outside the core to zero near the vortex center. The distance a over which ρ_s falls off to zero should be of the same order of magnitude as the effective core radius appearing in the expression of the line free energy,[9] and typically $a\sim 10$ Å. In the equations we may ignore small variations ρ_{s1} due to the incident wave, since they only appear in second-order terms outside the core (a usual approximation in acoustics), and they are negligible, in the core, compared to the inherent core variation $\rho_{s2}=\rho_s-\rho_{s0}$.

It is convenient to write $\vec{v}_s$ as the sum $\vec{v}_{s1}+\vec{v}_{s2}$, defining $\vec{v}_{s2}$ as the vortex flow. For a stationary vortex ($\vec{v}_{s1}=0$), $\vec{v}_{s2}=\vec{v}_{s2}{}^0(\vec{r})=(K/2\pi)(\vec{\nu}\times\vec{r}/r^2)$ and $\rho_{s2}=\rho_{s2}{}^0(r)$. In the presence of a second-sound wave, the vortex fields $\vec{v}_{s2}$ and ρ_{s2} are expected to be displaced (center at $\vec{r}_L$) *and* deformed. Nevertheless, we conjecture that the vortex fields are rigidly transported, supposing the deformation effects to be of second order in the linear response. In other words we will seek solutions of hydrodynamic equations, where $\vec{v}_{s2}=\vec{v}_{s2}{}^0[\vec{r}-\vec{r}_L(t)]$ and $\rho_{s2}=\rho_{s2}{}^0[\vec{r}-\vec{r}_L(t)]$; this implies the conditions

$$\begin{aligned}&\partial\vec{v}_{s2}/\partial t=-(\vec{v}_L\cdot\nabla)\vec{v}_{s2},\quad \partial\rho_s/\partial t=-\vec{v}_L\cdot\nabla\rho_s,\\&\nabla\cdot\vec{v}_{s2}=0,\quad \nabla\rho_s\cdot\vec{v}_{s2}=-\nabla\rho_n\cdot\vec{v}_{s2}=0\,.\end{aligned} \tag{5}$$

Such a drastic assumption will be justified later, in that we do succeed in finding a solution in the given form, satisfying the laws of mass and momentum conservation, and ultimately agreeing with experiment.

Because of its viscosity, the normal fluid is dragged around the moving vortex and we expect the normal flow $\vec{v}_n$ to be perturbed over distances of the order of the viscous penetration depth $\delta=(\eta/\rho_n\omega)^{1/2}$. At usual second-sound frequencies ($\omega\sim10^2$–10^4), and at small amplitudes (v_{n1}, $v_{s1}\sim 1$ mm sec^{-1}), the following conditions hold:

$$r_L,\ a\ll\delta\ll\lambda\,, \tag{6}$$

where λ is the second-sound wavelength. Let v be the order of magnitude of the velocities v_{n1}, v_{s1}, and v_L. From the condition $r_L\ll\delta$, or $v_L\ll\omega\delta$, it results that the nonlinear terms such as $(\vec{v}_n\cdot\nabla)\vec{v}_n\sim v^2/\delta$ in Eq. (4a) are negligible compared to the first term $|\partial\vec{v}_n/\partial t|\sim\omega v$. As $\lambda\gg\delta$, spatial variations of $\vec{v}_{n1}$ and $\vec{v}_{s1}$ are disregarded, so that $\nabla\cdot\vec{v}_{n1}=\nabla\vec{v}_{s1}=0$. Hence $\nabla\cdot\vec{v}_s=\nabla\cdot\vec{v}_{s2}=0$, and by the last condition (5), Eq. (3a) of mass conservation becomes

$$\rho_n\nabla\cdot\vec{v}_n-(\vec{v}_n-\vec{v}_{s1})\cdot\nabla\rho_s=0\,. \tag{3b}$$

Outside the core, $\nabla\rho_s=0$, so that Eq. (3b) reduces to $\nabla\cdot\vec{v}_n=0$. Inside the core, variations of $\vec{v}_n$ can be ignored, and so we may speak of its value $\vec{v}_n=\vec{v}_n(0)$ at the core. It amounts to neglecting terms of the order of $\rho_s v/\delta$ compared to terms of the order of $\rho_s v/a$ in Eq. (3b), which consequently reduces to $[\vec{v}_n(0)-\vec{v}_{s1}]\cdot\nabla\rho_s=0$. Since

$\nabla\rho_s \neq 0$ in the core, this implies that

$$\vec{v}_n(0) = \vec{v}_{s1}. \tag{7}$$

Now Eq. (4a) can be rewritten, allowing for the transport hypothesis (5) and the approximation conditions (6). Taking account also of Eq. (7), and using $\nabla\times\vec{v}_s = 0$, we find

$$\rho_n\,\partial\vec{v}_n/\partial t + \rho_s\,\partial\vec{v}_{s1}/\partial t + \tfrac{1}{2}\rho_s\nabla(v_{s2}{}^2) + \rho_s\nabla[(\vec{v}_{s1}-\vec{v}_L)\cdot\vec{v}_{s2}] + \vec{v}_{s2}[\nabla\rho_s\cdot(\vec{v}_{s1}-\vec{v}_L)] = -\nabla p + \eta\nabla^2\vec{v}_n. \tag{4b}$$

Note that the terms of second viscosity (ζ_1 and ζ_2)[8] in σ have disappeared because $\nabla\cdot\vec{j} = \nabla\cdot\vec{v}_n = 0$. Let us take the curl of both sides of Eq. (4b). Introducing $\vec{u} = u\vec{\nu} = \nabla\times\vec{v}_n$, we obtain:

$$\rho_{n0}\,\partial\vec{u}/\partial t - \eta\nabla^2\vec{u} = \nabla\times\vec{\varphi},$$
$$\vec{\varphi} = (\vec{v}_{s1}-\vec{v}_L)\times(\nabla\vec{\rho}_s\times\vec{v}_{s2}). \tag{8}$$

In the right side of Eq. (8), we have put together, in a condensed form, all the terms which differ from zero only in the core region, retaining only those of the order of $\rho_s v v_{s2}/a^2$. Two terms, $\nabla\rho_n \times\partial\vec{v}_n/\partial t$ and $\nabla\rho_s\times\partial v_{s1}/\partial t \sim \rho_s v\omega/a$, arising from the curl of the first two terms in Eq. (4b), are negligibly small since $\omega \ll v_{s2}/a \sim 10^8$ sec^{-1} [moreover, they cancel by Eq. (7)]. An even smaller term $|-\rho_{s2}\,\partial\vec{u}/\partial t| \sim \rho_s v\omega/\delta$, derived from the first term in Eq. (4b), has also been omitted.

At this stage, the problem of the dragging of the normal fluid is formally similar to the one of an ordinary viscous fluid submitted to a localized density of force $\vec{\varphi}$, the diffusion Eq. (8) following immediately in this case from the Navier-Stokes equation. In this respect, it is interesting to note that the total force $\vec{F}$ turns out to be independent of the unspecified ρ_s profile, and equal (except for the sign) to the classical Magnus force[2]

$$\vec{F} = \iint\vec{\varphi}\,d^2r = \frac{\rho_{s0}K}{2\pi}\,\vec{\nu}\times(\vec{v}_{s1}-\vec{v}_L). \tag{9}$$

As long as $r_L \ll \delta$, we may proceed with calculations by assuming $r_L \approx 0$, and approximate the source term of Eq. (8) linearly, $\vec{v}_L$, $\vec{\varphi}$, $\vec{u}$, as well as $\vec{v}_{s1}$, depending on time as $e^{i\omega t}$. Denoting the complex amplitudes of various quantities by the same symbols, let $U(\vec{k})$ and $\Phi(\vec{k})$ be the spatial Fourier transforms of $u(\vec{r})$ and $\varphi(\vec{r})$. From Eq. (8), we have

$$U(\vec{k}) = i\vec{k}\times\vec{\Phi}/(i\rho_{n0}\omega + \eta k^2). \tag{10}$$

For all values of $k \ll 1/a$ (or $1/r_L$), $e^{i\vec{k}\cdot\vec{r}} \approx 1$ throughout the core region, and therefore it follows that $\vec{\Phi} = \vec{F}/2\pi = \text{const}$. It is clear that, on the other hand, $\Phi(\vec{k}) = 0$ for $k \gg 1/a$. The precise behavior of $\Phi(\vec{k})$ for $k \sim 1/a$ obviously depends on the detailed structure of the vortex core. Pleading ignorance concerning the core structure, we shall use the approximate step function $\Phi(\vec{k}) = \vec{F}/2\pi = \text{const}$, up to $k = 1/l$, and 0 beyond $k = 1/l$, l being typically of the order of 1 Å if $a \sim 10$ Å. Such a procedure is justified because exact values of $\Phi(\vec{k})$ for large $k \sim 1/a$ are of little significance when calculating $u(\vec{r})$ from (10). As a matter of fact, the precise choice of the cut in the k space will not seriously affect the final result, l appearing in a logarithmic factor. Then, taking the Fourier transform of (10), we obtain

$$u(\vec{r}) = \frac{1}{2\pi}\iint U(\vec{k})e^{i\vec{k}\cdot\vec{r}}\,d^2k = \frac{\rho_{s0}K}{2\pi\eta}(\vec{v}_L-\vec{v}_{s1})\cdot\nabla\mathfrak{F},$$
$$\mathfrak{F}(r) = (1/2\pi)\iint_{0<k<1/l}\frac{e^{i\vec{k}\cdot\vec{r}}\,d^2k}{k^2+i/\delta^2}, \tag{11}$$

where $\mathfrak{F}(r)$ is an axially symmetrical function that we shall need explicitly only for $r = 0$. Since $\nabla\cdot\vec{v}_n = 0$ and $\nabla\times\vec{v}_n = \vec{u}$, we can find $\vec{v}_n$ from the vector $\vec{u}$ by using the Biot-Savart law

$$v_n(\vec{r}\,') = \vec{v}_{n1} + (1/2\pi)\iint d^2r\,\vec{u}\times(\vec{r}\,'-\vec{r})/|\vec{r}\,'-\vec{r}|^2. \tag{12}$$

Then, letting $\vec{r}\,' = 0$ in (12) and expressing the core condition (7), we obtain the required relation between $\vec{v}_L$, $\vec{v}_{n1}$, and $\vec{v}_{s1}$. A straightforward calculation yields

$$\vec{v}_n(0) = \vec{v}_{s1} = \vec{v}_{n1} + (\rho_{s0}K/4\pi\eta)\mathfrak{F}(0)\vec{\nu}\times(\vec{v}_L-\vec{v}_{s1}),$$
$$\mathfrak{F}(0) = \tfrac{1}{2}\ln(1-i\delta^2/l^2) \approx \ln\delta/l - i\pi/4.$$

This relation is of the general form (1) with $B' = 0$ and

$$B^{-1} = (\rho_n\rho_s K/16\pi\eta\rho)[\ln(\eta/\rho_n\omega l^2) - i\pi/2]. \tag{13}$$

Firstly, we note in this result the frequency-dependent term and the imaginary term, which appear in exactly the same form as in the H-V expression (2); both terms correspond to now well-established features of mutual friction, as indicated above. Secondly, if we compare the real part of B as given by Eq. (13) as a function of temperature with experimental values of B_1, we observe a very good agreement in the temperature range from 1.7 to 2.1 K, as shown in the

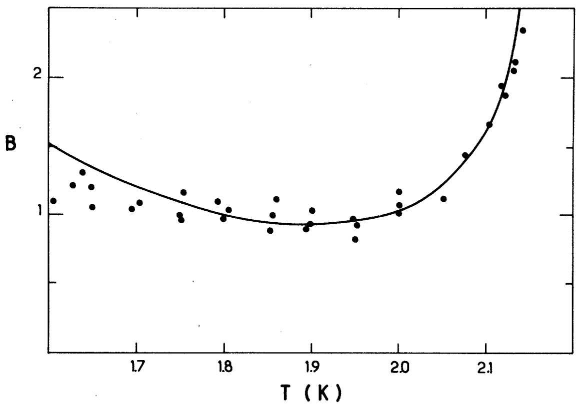

FIG. 1. The real part of the mutual-friction parameter B as a function of the temperature. Solid circles: Experimental data from Lucas (Ref. 4). Solid curve: Theoretical curve derived from Eq. (13).

Fig. 1. It is noteworthy that, in this range, B' is found experimentally to be nearly zero.[4] Outside this temperature interval, experimental values of B_1 are larger than expected from Eq. (13), while B' significantly deviates from zero. Agreement with experiment supports our model, clearly indicating that, at the temperatures concerned, the dissipative process in mutual friction is none other than the viscosity of the dragged normal fluid.

In the λ region, vortex dynamics must be reconsidered to include effects of relaxation of the superfluid density, which, according to Pitaevskii,[10] should become the principal energy-dissipation mechanism. In the low-temperature region, as pointed out above, we expect our hydrodynamic picture to fail, since $L \gg a$. Nevertheless, we suggest that the diffusion Eq. (8) may remain valid, as long as $L \ll \delta$; but the source term in Eq. (8), as well as the core condition (7), should be reexamined in the framework of a kinetic theory of rotons-vortex collisions, such as formulated by Goodman.[11]

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