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Chaos in a Laser System under a Modulated External Field

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It is shown that a single-mode laser under the influence of an external modulated field may show chaotic behavior. The power spectrum and the separation distance are calculated to demonstrate the existence of chaos.

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Recently, chaotic behaviors have been reported on various systems.[1-3] For a single-mode laser system Haken[4] showed that the laser equations reduce to the Lorenz equations with an appropriate scaling of variables. However, the realization of the Lorenz-type chaos in the laser system is difficult because of some restrictions on parameters. Graham[5] showed that a reduction to the Lorenz equations may also be obtained for a mode-locked pulse train in an infinite laser system or in a ring laser, but the periodic boundary conditions in the latter case select periodic solutions only. For a laser system under an external field, Rabinovich reported that a chaotic behavior appears for a certain range of parameters.[6] However, for values inside the region of parameter space quoted in Ref. 6, we have not been able to reproduce his chaotic state. In the present note, chaos is numerically shown to exist in a laser system under a modulated external field.

We will use the approximation of a spatially-homogeneous field and assume single-mode operation. For simplicity we will assume that the resonance frequency of the two-level atoms and the cavity frequency are equal. Then the laser equations read[7]

$$\begin{aligned} dE/dt &= -\kappa(E - E_{\text{ext}}) + igP, \\ dP/dt &= -\gamma_\perp P - igE\sigma, \\ d\sigma/dt &= -\gamma_\parallel(\sigma-\sigma_0) - 2ig(PE^* - P^*E), \end{aligned} \tag{1}$$

where E, P, and σ are the complex light amplitude, the total complex dipole moment and the inversion, respectively. We consider a laser system under a time-dependent external field E_{ext}. We approximate Eq. (1) by assuming $\kappa \ll \gamma_\perp, \gamma_\parallel$. Then the adiabatic elimination of the atomic variables, P and σ, yields

$$dx/dt = -i\Omega x + (z-1)x + A(\tau) \tag{2}$$

with $z = R/(1+|x|^2)$, where we put, $t = \tau/\kappa$, $R = g^2\sigma_0/\gamma_\perp$, $E = (\gamma_\perp\gamma_\parallel)^{1/2}x\exp(i\Omega\tau)/2g$, $E_{\text{ext}} = (\gamma_\perp\gamma_\parallel)^{1/2} \times A(\tau)\exp(i\Omega\tau)/2g$. The parameter Ω is the detuning of the external field frequency from the cavity frequency. The modulation of the external field is represented by $A(\tau)$. We will first consider the case, $A(\tau) = a$ (= const). A steady state of Eq. (2) is obtained by putting the right-hand side equal to 0. We denote the steady state-value of z as z_s. This steady state loses its stability when a root of the equation,

$$\lambda^2 - 2\lambda\frac{z_s^2 - R}{R} + \frac{z_s}{R}[(z_s-1)^2 + \Omega^2]\frac{\delta R}{\delta z_s} = 0, \tag{3}$$

has a positive value, where z_s satisfies

$$R = z_s + a^2 z_s/[(z_s-1)^2 + \Omega^2]. \tag{4}$$

From Eqs. (3) and (4) it may be readily seen that for a sufficiently large R the steady state always becomes unstable. For small a the instability occurs at $R = 1 + 2a^2/\Omega^2$. The instability is of hard-mode type. Therefore, we can expect that above a certain value of R the time evolution of Eq. (2) shows a limit-cycle behavior. The phase diagram of Eq. (2) is shown in Fig. 1. Above (below) each curve the limit cycle (the steady state) ap-

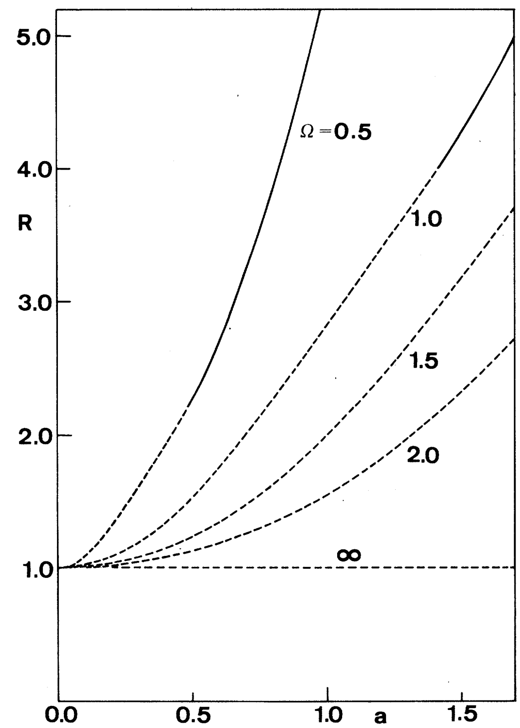

FIG. 1. The phase diagram for nonmodulated external field. Each curve corresponds to a different value of Ω. Above each curve, a limit cycle appears. The type of the transition from the steady state to the limit cycle is of the second order (dashed line) or of the first order (solid line).

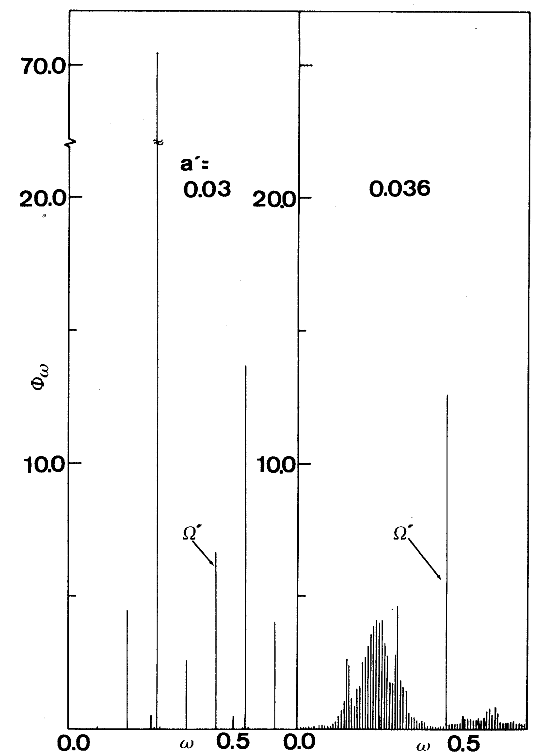

FIG. 2. The power spectrum Φ_ω of the periodic ($a' = 0.03$, the left figure) and the chaotic ($a' = 0.036$, the right figure) states. The sharp peaks at the frequency $\omega = 0.45$ in both figures correspond to the frequency of the external modulated amplitude. The resolution is $2\pi/(1024\times 30\times 0.025)$. The average is taken over the sequence of the spectrum 50 times.

pears. In other words, for fixed Ω and parameters corresponding to the region above the curve the electric field is spontaneously modulated although the incident electric field has a constant amplitude.

Now let us turn to the case of the modulated external field. The modulation amplitude $A(\tau)$ is assumed to have the following form,

$$A(\tau) = a + a'\cos(\Omega'\tau). \tag{5}$$

To proceed further it is useful to choose a particular set of parameters. We adopt here $R = 2.0$, $\Omega = 0.5$, $a = 0.4$, and $\Omega' = 0.45$. We have numerically studied the time evolution of Eq. (2) by 4 methods: (1) Poincaré map, (2) Lorenz map, (3) power spectrum, and (4) separation distance. We have adopted a modification of the numerical integration method given in Lorenz's work.[1] When $a' = 0$ the reference system shows a limit-cycle behavior with the angular velocity $\Omega_0 = 0.2714$. As a' increases, the system behaves quasiperiodically with two characteristic frequencies, Ω' and Ω_0. By increasing a' further the limit cycle is entrained by the external force $A(\tau)$. As $\Omega_0/\Omega' \cong 0.6031 \cong \frac{3}{5}$, the entrainment occurs at a rational frequency of Ω', $\frac{3}{5}\Omega'$. Therefore, when we observe the time evolution at time intervals $2\pi/\Omega'$, a quintuple cycle (we will use this terminology hereafter) is seen to be realized. This periodic state loses its stability at $a' \approx 0.339$ to lead to a chaotic state. The power spectra of the periodic and the chaotic states are shown in Fig. 2. A broad peak is clearly seen in the chaotic state. To verify this behavior we plot the separation distance of two initially adjacent points. The method is the following: In an aged system (after a large number of steps the phase point can be considered to be trapped in the attractor) we consider a phase point and choose another point which is separated from this point by a small distance. In the present case the real part of x is chosen separated by the distance 0.000 01. Then the dis-

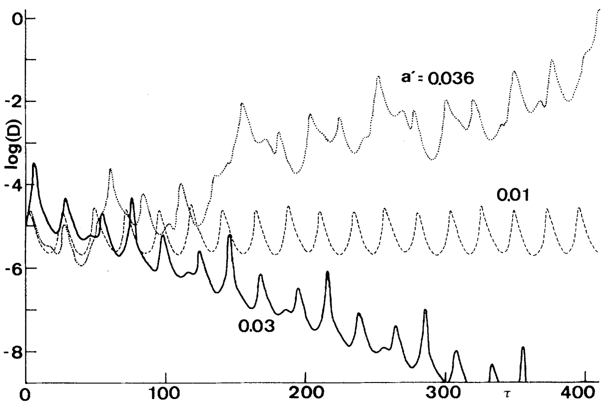

FIG. 3. The separation distance $D(\tau)$ of the two initially adjacent points against the time τ. The logarithmic value of $D(\tau)$ is plotted. Three curves correspond to the quasiperiodic ($a' = 0.01$), the periodic ($a' = 0.03$) and the chaotic ($a' = 0.036$) states, respectively.

tance, $D(\tau)$, between these two points is plotted in Fig. 3 versus the time τ. In the quasiperiodic state ($a' = 0.01$) it can be seen that the two phase points remain close to each other. In the periodic state ($a' = 0.03$) the phase points approach one another as the system evolves. The reason is that this periodic state appears due to the entrainment of the phase point by the external field, and the relative phase of the phase point to the external force $A(\tau)$ becomes fixed on the attractor. Therefore, the two phase points coincide with each other as $\tau \to \infty$. On the other hand in the chaotic state the two phase points get more separated as time goes on. The saturation behavior appears after $\tau \sim 400$. This is due to the fact that the size of the strange attractor (in the present case it is of the order one) is finite. This behavior of $D(\tau)$ is quite in line with the other examples of chaos.[8]

At sufficiently large a' ($\gtrsim 0.15$), the time evolution of the system is periodic with the frequency Ω'. Between this completely entrained state and the chaos mentioned above there appear various states.[9] The bifurcation scheme shows a window structure resembling the structure found by Tomita and Kai.[3] For example, the system has a octuple periodic state at $a' = 0.05$ and a chaotic state at $a' = 0.07$. The detailed bifurcation scheme with the variation of a' as well as with that of Ω' is interesting, but is beyond the scope of this short note.

If the set of parameters, Ω, a, and R, is chosen such that the system is deep inside the limit-cycle region, it becomes harder to find chaos. The reason may be that near the transition region between the steady state and the limit-cycle state the orbit of limit cycle is easily affected by the external force, while deep inside the limit-cycle region a strong modulation of $A(\tau)$ is necessary to change the limit-cycle orbit and it may violate the inequality $a > a'$.

The existence of chaos studied in the present note seems not to depend critically on the particular approximation (the adiabatic approximation) made at the beginning. Sufficiently close to the transition region between the steady state and the limit-cycle states we can always expect to get a bifurcation scheme leading to chaos if only we choose appropriate values of the parameters to reach the chaotic state.

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