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Transient Fluctuations in the Decay of an Unstable State

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The decay of an unstable state by the stochastic time of intersection of a given threshold is described. This stochastic parameter is characterized by closed moment equations exactly soluble, whereas previous approaches, in terms of stochastic amplitudes sampled at a fixed time, led to an open hierarchy of equations requiring approximate solutions. The relevance of this new description is shown by the fit with the experimental transient fluctuations of an unstable system.

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A nonequilibrium system, under the action of external parameters, may undergo transitions in the sense that one (or a set of) of its macroscopic observables have a sizable change. Usually these changes have been studied by a slow setting of the external parameter, in order to measure the stationary fluctuations and their associated spectra around each equilibrium point. A classification of these transitions is underway,[1] by an extensive use of sophisticated measurements[2] or computer experiments.[3] In particular, in quantum optics one can perform accurate statistical measurements by photon counting statistics; this method has been used to explore quantum optical transitions.[4]

More dramatic evidence, as well as detailed information, on the decay of an unstable state, and its leading to multiple nearby stable positions with different branching probabilities, can be obtained by applying sudden jumps to the driving parameter and observing the statistical transients.[5] These should by no means be compared with stationary time correlations (or their frequency spectra), since a linear regression is no longer valid, with the nonlinearities playing their full role. Furthermore, when the system is prepared in an unstable state, no net systematic forces are applied on its observables, and the decay is initiated by microscopic fluctuations. In the first linear part of the decay process the fluctuations are amplified; hence during the transient, and until nonlinear saturation near the new stable point reduces them, fluctuations do not scale with the reciprocal of the system size, as it is at equilibrium.

A first experiment on the photon statistics of the laser field during its switch on[5] has opened the discussion on transient statistics and the associated anomalous fluctuations.

Limiting for simplicity the discussion to the case of one stochastic amplitude x, the most natural experimental approach was to measure the probability density $P(x, t)$ at a given time t after the sudden jump of the driving parameter. Under general assumptions, $P(x, t)$ can be shown to obey a nonlinear Fokker-Planck equation (FPE). A time-dependent solution of the FPE in terms of an eigenfunction expansion is unsuitable for the large number of terms involved in the summation, with the exception of small jumps near threshold[6] or the asymptotic behavior for long times.[7]

Solving for the moments $\langle x^k(t)\rangle$ leads to an open hierarchy of coupled equations. A two-piece approximation first introduced for the laser[8] and then extended to other cases[9,10] consists in first letting the system decay from the unstable point under the linearized part of the deterministic force, diffusing simultaneously because of the stochastic forces. This leads to a short time probability distribution of easy evaluation. Then we solve for the nonlinear deterministic path and spread it over the ensemble of initial conditions previously evaluated in the linear regime.

A recent nonpiecewise treatment consisted in a $1/N$ expansion of the diffusion term (N being the system size).[11] However, this approximation fails for small jumps above the threshold of instability or for nonlinear diffusion coefficients.

Another approach[12] was to trace back at any time a virtual ensemble of initial conditions, which, inserted in the noise-free dynamic equations, would be responsible for the actual spread. This approach reduces the FPE to a diffusion equation; however, it fails for large deviations from the Gaussian as shown in a recent generalization.[13]

Here we present an exact approach to transient statistics which overcomes the limitations of the previous treatments. In this approach, we consider the time t at which a given threshold z_F is crossed as the stochastic parameter, whose distribution $Q(t, z, z_F)$ in terms of the interval between the initial position z and z_F must be assigned. This way, the time is no longer an ordering parameter to classify the sequence of measurements, but an interval limited by a start-stop operation between the onset of the instability and the passage through an assigned amplitude value.

Let $P(x, t)$, with

$$\int_{-\infty}^{\infty} P(x, t)\,dx = 1, \tag{1}$$

be the instantaneous probability density for the amplitude x which gets unstable under a force $F(x)$ and a noise delta correlated with a correlation amplitude $D(x)$. $P(x, t)$ is solution of the FPE

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[F(x)P] + \frac{\partial^2}{\partial x^2}[D(x)P]. \tag{2}$$

In order to develop an equation for the new density $Q(t, z)$ the time must be assigned as a single value parameter of z. This amounts to considering the problem of the first passage time in Brownian motion, which is ruled by the Kolmogorov equation[14,15]

$$\frac{\partial Q}{\partial t} = F(z)\frac{\partial Q}{\partial z} + D(z)\frac{\partial^2 Q}{\partial z^2}, \tag{3}$$

where z is the initial value ($t=0$), and the normalization is

$$\int_0^{\infty} Q(t, z)\,dt = 1. \tag{4}$$

Since we are studying the space evolution of the time distribution, Eq. (3) must be considered as a second-order differential equation and we need two boundary conditions: (i) the final value (the threshold) z_F, and (ii) the value α above which the process has to remain limited during the evolution.[16] Like for the usual FPE the evaluation of the moments is formally equivalent to the solution of the equation, but in this case we can in principle evaluate exactly all the moments by means of a recurrence formula. If

$$T_m \equiv \int_0^{\infty} t^m Q(z, t)\,dt, \tag{5}$$

we get

$$F(z)T_m' + D(z)T_m'' = -mT_{m-1} \tag{6}$$

(the prime denoting differentiation with respect to z). Thus, this way of treating the stochastic transient leads to a recursive set of moment equations exactly soluble. In particular, we have for the mean time T_1

$$T_1(z) = \int_z^{z_F}[dy/W(y)]\int_{-\infty}^{y} dx\,W(x)/D(x), \tag{7}$$

where we have defined

$$W(x) \equiv \exp\int_{x_0}^{x} d\xi\, F(\xi)/D(\xi); \tag{8}$$

Eq. (7) holds for $z_F > z$ and $\alpha = -\infty$. For a spread in the initial position z, $T_1(z)$ should be still averaged over the set of z. In a similar way we obtain

$$T_2(z) = \int_z^{z_F}\frac{dy}{W(y)}\int_{-\infty}^{y} dx\,\frac{2T_1(x)}{D(x)}W(x). \tag{9}$$

When we apply this formalism to the decay of unstable states, since D scales with the inverse system size, we can expand the above results in D series and display the first relevant correction to the deterministic solution. We find immediately that

$$T_1(z) = \int_z^{z_F}\frac{dy}{F(y)} + \int_z^{z_F} dy\,D\,\frac{dF/dy}{F^3}, \tag{10}$$

where the first term on the right-hand side is the deterministic part. Similarly, performing the approximation for T_2 we obtain for the variance $\Delta T \equiv T_2 - T_1^2$ the following relation

$$\Delta T = 2\int_{z}^{z_F} dy\, D(y)/F^3(y). \tag{11}$$

The method here presented overcomes the drawbacks of the previous approximate treatments.[5-13] It allows an exact treatment of instabilities consequent to a sudden jump in the control parameter, as laser transients,[5,8] superfluorescence,[17,18] and spinodal decomposition.[19]

In order to show the power of this approach, we have measured the crossing time probability distributions for an electronic oscillator driven from below to above threshold.[20]

Figure 1 gives the mean oscillator amplitude and its variance versus time as in the usual stochastic treatment of transients.[5] Figure 2 gives the variance of crossing times for increasing thresholds as defined here. The theoretical fit of the experimental points is an application of Eqs. (10) and (11), as briefly discussed in the following.

A nonlinear oscillator suddenly driven from below to above threshold at $t=0$ can be described by the deterministic equations for the amplitude x[21]

$$\dot{x} = F_-(x) = -a_1 x, \quad t<0; \tag{12a}$$

$$\dot{x} = F_+(x) = a_2 x - b x^3, \quad t>0; \tag{12b}$$

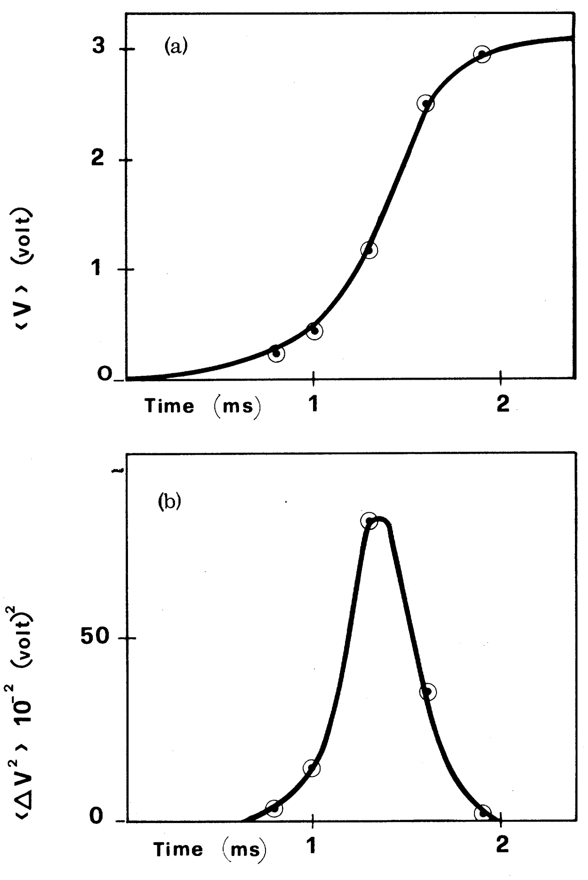

FIG. 1. Transient statistical evolution of an electronic oscillator driven from below to above threshold by a sudden jump. No external noise added. Oscillator parameters: $a_1 = 2.4\times10^2$ s^{-1}, $a_2 = 3.2\times10^3$ s^{-1}, $b = 3.5\times10^2$ V^{-2} s^{-1}, $D = 0.18$ V^2 s^{-1}. (a) Evolution of the average amplitude $\langle V\rangle$. (b) Evolution of the variance $\langle\Delta V^2\rangle$. Dots, experimental; line, theory.

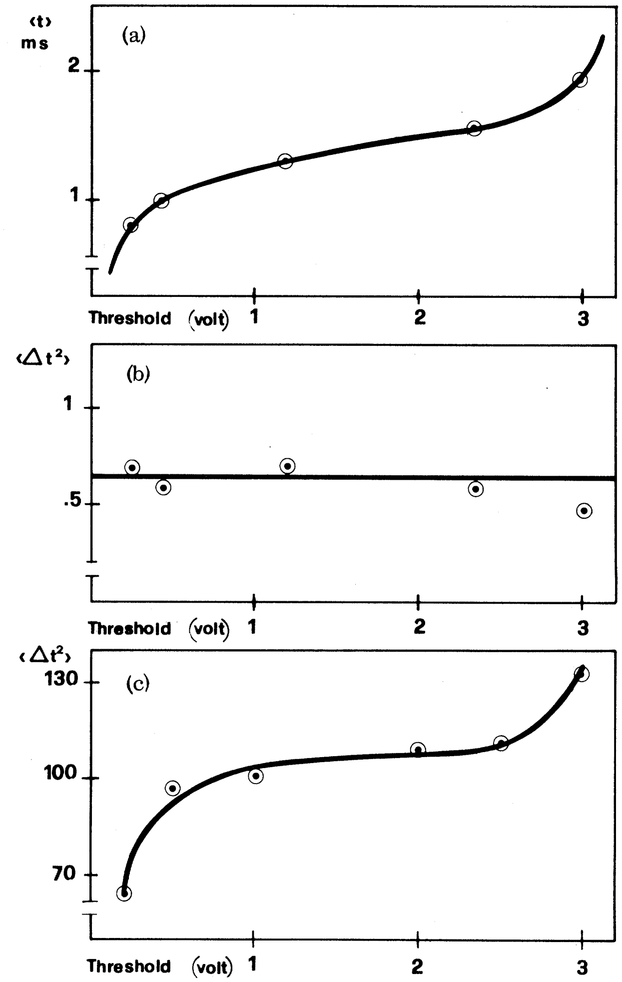

FIG. 2. Transient oscillator as in Fig. 1. Statistical distribution of the time intervals between the initial condition and the crossing of the threshold V. (a) Average crossing time $\langle t\rangle$; (b) variance $\langle\Delta t^2\rangle$ under the action of the internal noise ($D = 0.18$ V^2 s^{-1}); (c) variance $\langle\Delta t^2\rangle$ for an added external noise ($D = 8.6$ V^2 s^{-1}). In (b) and (c) the scale is in units of 10^{-9} s^2. Dots, experimental; line, theory.

where $a_1, a_2 > 0$ are the linear loss and gain rates, and $b > 0$ weights the first relevant nonlinearity.

Furthermore, the oscillator is perturbed by a white noise force of correlation intensity $2D = 2\epsilon d$, where ϵ is a smallness parameter of the order of inverse system size N.

The amplitude x starts from an initial Rayleigh distribution with a mean value $\pi^{1/2}/2\ (D/a_1)^{1/2}$ to a final value $(a_2/b)^{1/2}$ with a residual spread of rms $(D/a_2)^{1/2}$.

The introduction of the force of Eq. (12b) in the FPE (2) or the use of the standard piecewise approximation[8-10] yields the theoretical fits of the experimental points of Fig. 1.

Insertion of Eqs. (12) into Eqs. (10) and (11) plus averaging over the initial z distribution leads to the theoretical fits of Fig. 2. An improved fit[20] is obtained by the evaluation of the complete Eqs. (7) and (9). In Fig. 2 we give the variance in time crossings due to the internal noise (which scales as $1/N$), and the increased variance due to the application of a large external noise of amplitude D_0.

The following comments convey some of the relevant physics: (i) The first term of Eq. (10) yields an average decay time which scales as $\langle T_1\rangle \sim \ln(a_1 N)$, that is, the logarithmic divergence with the system size N can be compensated by an initial preparation close to threshold so that $a_1 N$ remains finite. (ii) For large systems ($N \to \infty$) the variance ΔT does not depend on z_F and is approximately given by

$$\Delta T \sim a_1/4\pi a_2^3. \tag{13}$$

A constant variance for increasing thresholds means that the various trajectories are shifted versions of the same deterministic curve, and the noise scaling as $1/N$ plays a role only in spreading the initial condition. (iii) The introduction of an external noise D_0 adds a fluctuation peculiar for each path, giving a ΔT dependent on z_F.

In conclusion, we have shown for the first time a clear separation between the role of the initial spread and the noise along each path, and have introduced a new experimental characterization of a statistical transient which can be dealt with in an exact way. A first passage method has also been used recently to evaluate the mean tunneling time between the two valleys of a bimodal potential.[22] Our approach, however, includes higher-order moments.

A forthcoming paper[20] exploits this method for a general classification of time-dependent effects in experimental unstable systems.

As here presented, the method seems limited to discrete variables and not applicable to field problems as diffusive instabilities in hydrodynamics or chemical reactions. However, by a suitable mode expansion and selection of the lowest-threshold modes[23] one can reduce field problems to a set of a few discrete coupled variables which can be dealt with by our method.

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