# Dimension of Strange Attractors

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A relationship between the Lyapunov numbers of a map with a strange attractor and the dimension of the strange attractor has recently been conjectured. Here, the conjecture is numerically tested with use of several different maps, one of which results from a system of ordinary differential equations occurring in plasma physics. For the cases tested, the conjecture is verified to within the obtained accuracy.

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An *attractor* is a subspace of some ordinary $N$-dimensional space to which the solution of an $N$-dimensional dynamical system of equations asymptotes for large time. Two cases of dynamical systems will be considered here: maps (discrete time variable $j$),

$$\vec{x}_{j+1}=\vec{f}(\vec{x}_j), \tag{1}$$

where $j$ is an integer, and autonomous ordinary differential equations (continuous time),

$$d\vec{X}(t)/dt=\vec{F}(\vec{X}), \tag{2}$$

where $\vec{F}$, $\vec{f}$, $\vec{X}$, and $\vec{x}$ are $N$-dimensional vectors. Equation (1) generates a sequence $\vec{x}_1, \vec{x}_2, \ldots$ if an initial $\vec{x}_1$ is given, while Eq. (2) generates an orbit $\vec{X}(t)$, if $\vec{X}(0)$ is given. A *strange* attractor may, for most purposes, be thought of as an attractor with dimension $d<N$, where $d$ is noninteger. The relevant definition of dimension is that due to Hausdorff[1]

$$d=\lim_{\epsilon\to 0}[\ln n(\epsilon)][\ln(\epsilon^{-1})]^{-1}, \tag{3}$$

where $n(\epsilon)$ is the number of $N$-dimensional cubes of side $\epsilon$ needed to cover the attracting subset. Alternatively, $n(\epsilon)\cong K\epsilon^{-d}$ for small $\epsilon$, where $K$ is a constant.

Strange attractors have received special attention in recent years because of the possibility that they occur in a wide variety of physical situations. An interesting attribute of strange attractors in that they lead to chaotic or turbulent orbits. For example, the onset of turbulence in fluids in currently thought to coincide with the appearance of a strange attractor.[2]

A possible reason for interest in the Hausdorf dimension of a strange attractor is that it says something about the amount of information necessary to specify the attracting set to within an accuracy $\epsilon$. More concretely, if one wanted to give a coarse-grained distribution function $\hat{f}_\epsilon(\vec{X})$ for the approximate calculation of a time average over the turbulent evolution of a given function $g$ of $\vec{X}(t)$, then one would write

$$\langle g(\vec{X})\rangle \cong \int \hat{f}_\epsilon(\vec{X})g(\vec{X})\,d^NX, \tag{4}$$

where $\langle g(\vec{X})\rangle$ is the time average of $g(\vec{X}(t))$, $\epsilon$ is the coarse-graining scale, and (4) results from the ergodic hypothesis if $g(\vec{X})$ is assumed to be slowly varying in the scale $\epsilon$. One way of constructing $\hat{f}_\epsilon$ is to divide the original $N$-dimensional space into cubes of side $\epsilon$ and then specify the fraction of time that the orbit on the strange attractor spends in each cube. Only $n(\epsilon)$ cubes will have nonzero values of $\hat{f}_\epsilon$. Thus, the information necessary to specify $\hat{f}_\epsilon$ is the coordinates of the $n(\epsilon)$ cubes in which $\hat{f}_\epsilon \neq 0$ and the value of $\hat{f}_\epsilon$ in each of these cubes. Hence, in principle, the information necessary to specify $\hat{f}_\epsilon$ is contained in $n(\epsilon)(N+1)$ numbers. Thus, the dimension says something about the amount of information necessary to characterize the attractor.

Recently, a relationship between the dimension of a strange attractor of an $N$-dimensional map [Eq. (1)] and the *Lyapunov numbers* of the map has been conjectured.[3] Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the Lyapunov numbers of the map ordered so that $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_N$. Then Kaplan and Yorke[3] conjecture that (the result of Mori and Fujisaka[3] is different for $N>2$)

$$d=j+[\ln(\lambda_1\lambda_2\cdots\lambda_j)][\ln\lambda_{j+1}^{-1}]^{-1}, \tag{5}$$

where $j$ is the largest number for which $\lambda_1\lambda_2\cdots\lambda_j$

 

$>1$. The Lyapunov numbers $\lambda_i$ are defined to be

$$\lambda_i = \lim_{q\to\infty} \{\text{magnitude of the eigenvalues of } \underline{J}(\vec{x}_q)\cdots\underline{J}(\vec{x}_2)\underline{J}(\vec{x}_1)\}^{1/q}, \quad (6)$$

where $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_q$ is an orbit generated by (1), and $\underline{J}(\vec{x})$ is the Jacobian matrix of (1), $J_{ij}(\vec{x}) = \partial f_i(\vec{x})/\partial x_j$. For the special case $N=2$ with $\lambda_1 > 1 > \lambda_1\lambda_2$, Eq. (5) becomes

$$d = 1 + (\ln\lambda_1)/(\ln\lambda_2^{-1}). \quad (7)$$

As a way to intuitively motivate Eq. (7), we have constructed a simple special map for which (7) is satisfied exactly,

$$x_{n+1} = \lambda_2 x_n + y_n - \lambda_1^{-1} y_{n+1},$$

$$y_{n+1} = \lambda_1 y_n \pmod 1, \quad (8)$$

where $\lambda_1 > 1 > \lambda_1\lambda_2 > 0$ is assumed and we take $\lambda_1$ to be an integer. This map may be viewed as resulting from two operations, illustrated in Fig. 1 for $\lambda_1 = 3$, which map the unit square, $0 \le x \le 1$, $0 \le y \le 1$, into itself. Application of (8) $M$ times will map the unit square into $\lambda_1^M$ vertical bands each of width along $x$ of $\lambda_2^M$. Furthermore, it can be shown that these $\lambda_1^M$ bands are contained within the $\lambda_1^{(M-1)}$ bands that result from $M-1$ applications of (8) to the unit square. Clearly the dimension of the attractor along $y$ is 1. The dimension along $x$ can be obtained from Eq. (3) by noting that the necessary number of coverings of length $\epsilon = \lambda_2^p$ is $\lambda_1^p$ ($p$ is an integer). Equation (7) then follows.

In order to apply Eq. (5) to the case of ordinary differential equations, Eq. (2), we introduce the *Lyapunov exponents*, $h_1, h_2, \ldots, h_N$, where $N$ denotes the dimension of the system (2). Viewing the ordinary differential equations as generating a map advancing $\vec{X}$ forward by some fixed arbitrary increment in time, $\tau$, we can identify $\lambda_i = \exp(h_i\tau)$, and insert the $\lambda_i$ in (5). The result is independent of $\tau$. For example, for $N=3$, we

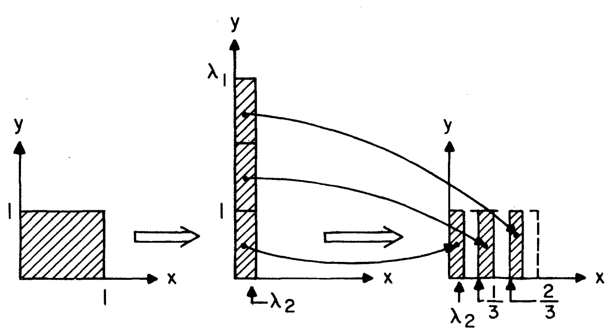


FIG. 1. Illustration of the map Eq. (8) for $\lambda_1 = 3$.

have (see also Mori[3])

$$d = 2 - h_1/h_3, \quad (9)$$

where we have assumed $h_1 > 0 > h_3$ and $h_1 + h_3 < 0$ and made use of the fact that[4] $h_2 = 0$.

In what follows, we will first describe some numerical experiments designed to test Eq. (7). The technique will then be applied to a test of Eq. (9) with use of a system of ordinary differential equations.[5]

The Hausdorf dimension for a strange attractor of a two-dimensional map is calculated using a computer program based on Eq. (3). To calculate $n(\epsilon)$ the space is divided into boxes of side $\epsilon$. An initial vector is chosen, and the map is iterated a sufficient number of times (i.e., much greater than $\ln\epsilon/\ln\lambda_2$) that the subsequently generated points can be considered to be on the attractor. A list is made of those boxes containing at least one point on the attractor. Each newly generated point on the attractor is checked to see if its box is on the list. If not, it is added to the list. After many iterations, the number of boxes on the list approaches $n(\epsilon)$. For small $\epsilon$, we expect that $n(\epsilon) \sim K\epsilon^{-d}$. Thus, defining $d_\epsilon \equiv \{\ln[n(\epsilon)]\}\{\ln\epsilon^{-1}\}^{-1}$, we see that $d_\epsilon - d \cong (\ln K)/(\ln\epsilon^{-1})$. It is difficult to make $d_\epsilon - d$ small by making $\epsilon$ small since the dependence is logarithmic. Note, however, that for small $\epsilon$ a plot of $d_\epsilon$ vs $[\ln\epsilon^{-1}]^{-1}$ will be approximately linear. This is, in fact, observed numerically. Our "measured" values of $d$ are determined by least-squares fitting a straight line to $d_\epsilon$ vs $[\ln\epsilon^{-1}]^{-1}$ for several small values of $\epsilon$, and then extrapolating the result to $\epsilon \to 0$. The accuracy of the result is estimated from the standard deviation of the points from the fitted line. The above-described dimension measuring program has been tested, with good results by use of several sets of known dimension [an area, a line, a Cantor set, and Eqs. (8), among others].

Tests of Eq. (7) with use of three different two-dimensional maps were performed. The three maps are one originally studied by Henon[6] ($x_{j+1} = y_j + 1 - a x_j^2$, $y_{j+1} = b x_j$), one introduced by Kaplan and Yorke[3] [$x_{j+1} = 2x_j \pmod 1$, $y_{j+1} = \alpha y_j + \cos 4\pi x_j$], and one studied by Zaslavskii[7] as a model of the effect of dissipation on a Hamiltonian system $\{x_{j+1} = [x_j + \nu(1 + \mu y_j) + \epsilon\nu\mu\cos 2\pi x_j] \pmod 1$, $y_{j+1} = \exp(-\Gamma)(y_j + \epsilon\cos 2\pi x_j)$, where $\mu$

TABLE I. Summary of test data.

| System tested | $d$ from Lyapunov numbers | $d$ from program based on Eq. (3) |
|---|---|---|
| Henon map, $a = 1.2$, $b = 0.3$ | $1.200 \pm 0.001$ | $1.202 \pm 0.003$ |
| Henon map, $a = 1.4$, $b = 0.3$ | $1.264 \pm 0.002$ | $1.261 \pm 0.003$ |
| Kaplan and Yorke map, $\alpha = 0.2$ | 1.430 676 6 | $1.4316 \pm 0.0016$ |
| Zaslavskii map, $\Gamma = 3.0$, $\epsilon = 0.3$, $\nu = 10^2 \times 4/3$ | $1.387 \pm 0.001$ | $1.380 \pm 0.007$ |
| Ordinary differential equations of Ref. 5 | $2.317 \pm 0.001$ | $2.318 \pm 0.002$ |

$\equiv [1 - \exp(-\Gamma)]\Gamma^{-1}\}$. For all of these maps $\det \underline{J}(\vec{x})$ is a constant independent of $\vec{x}$ [$-b$ for the Henon map, $2\alpha$ for the Kaplan-Yorke map, and $\exp(-\Gamma)$ for the Zaslavskii map]. Thus, all of these maps lead to uniform contraction of areas on each iteration, and $\lambda_1\lambda_2 = |\det \underline{J}(\vec{x})|$. Furthermore, for the map of Kaplan and Yorke the Lyapunov numbers may be calculated analytically, $\lambda_1 = 2$ and $\lambda_2 = \alpha$. To find the Lyapunov numbers for the Henon and Zaslavskii maps we utilize Eq. (6) to calculate the largest eigenvalue, $\lambda_1$, and then find $\lambda_2$ from $\lambda_1\lambda_2 = |\det \underline{J}(\vec{x})|$. [$\lambda_2$ is usually inaccurately determined from (6) unless a very large number of decimal places is retained in the calculation.]

The first four rows of Table I summarize results from tests with use of the above-mentioned two-dimensional maps. The second column is the value of $d$ predicted from Eq. (7). The third column gives the value of $d$ calculated using our dimension measuring program. The fifth row of the table gives results of a similar test of Eq. (9) with use of a system of three ordinary differential equations that describes the saturation of a linearly unstable plasma wave via cubicly nonlinear coupling to linearly damped waves.[5,8] For the system studied, the divergence of $\vec{F}(\vec{X})$ [cf. Eq. (2)] is a negative constant independent of $\vec{X}$, $\partial F_1/\partial X_1 + \partial F_2/\partial X_2 + \partial F_3/\partial X_3 = -k$. Thus, by the divergence theorem, phase-space volumes evolved according to the given system of equations will shrink exponentially in time, and $h_1 + h_3 = -k$. Thus, Eq. (9) yields $d = 2 + h_1(h_1 + k)^{-1}$. $h_1$ can be determined by numerically computing the average exponential divergence of two infinitesimally close-by points. Thus, we obtain a prediction of $d$. To compute $d$ based on Eq. (3), we associate a Poincaré map with the differential equation system.[5] This gives a picture of the intersection of the strange attractor with a surface (the "surface of section"). The dimension-measuring program is applied to this intersection. The dimension of the strange attractor is then the dimension of the intersection plus one. The last row of Table I compares results from this procedure with $d = 2 + h_1(h_1 + k)^{-1}$.

It is evident from Table I that the predicted and measured values of $d$ agree to within the accuracy obtained for all cases considered. We note that calculation of the dimension from the Lyapunov numbers is computationally much less costly than using a routine based upon Eq. (3). As an example, for the second row of Table I, the calculation of the measured value of $d$ required about 5 min on the Cray computer and about $4 \times 10^5$ words of memory. The calculation of $\lambda_1$, however, required about 0.3 min, and a relatively insignificant amount of memory.

Based on further development of the theory,[9] it is now clear that (7) and (9) cannot be true in general. In particular, they should not be expected to hold for cases where the eigenvalues of the Jacobian depend on $\vec{x}$ (i.e., all the maps we have tested except for that of Kaplan and Yorke). The results in Table I, however, indicate that, even if (7) and (9) do not hold exactly, they must still yield a surprisingly good approximation to the dimension for typical cases where the contraction rate is constant. The reason for this close agreement is currently under investigation.

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