$Q^2) \sim (1/Q^2)(\ln Q^2/\Lambda^2)^{-\gamma n} A_n(M^2)$ but no new M^2/Q^2 correction terms develop.

[8]See W. Frazer and J. Gunion, Phys. Rev. D 20, 147 (1979), and references therein.

[9]S. Brodsky and P. Lepage, Phys. Lett. 43, 545 (1979).

[10]The diagram in Fig. 3 gives rise to a gauge-invariant calculation of F_3 even without including other diagrams in which the hadronic vertex is probed. Such diagrams do, however, add to the corrections given in (17). See Ref. 5.

[11]In the limit $\sigma << M^2$ the x dependence of the structure function is highly unphysical. In any case a reanalysis of (15) in this limit yields multiple M^2/Q^2 corrections to M_3^n for $n \geq 3$ (see Ref. 5).

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Gluon Condensation from Trace Anomaly in Quantum Chromodynamics

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Condensation of the operator $(G_{\mu\nu}{}^a)^2$ in quantum chromodynamics is shown by constructing the effective potential through the trace anomaly equation. Effects of Wilson loop on the condensation are also studied.

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Certainly one of the most important problems in low-energy quantum chromodynamics (QCD) is the determination of its correct ground state. Various pictures have been offered[1] as to how it might be formed, but at the moment we do not yet have secure knowledge of its structure and the primary mechanism(s) responsible for it. In this note we study the structure of the vacuum in QCD with massless quarks by constructing the effective potential for the gauge-invariant gluonic operator $\hat{\varphi} \equiv \frac{1}{4}\int d^4x [G_{\mu\nu}{}^a(x)]^2$, where $\hat{G}_{\mu\nu}{}^a \equiv \partial_\mu \hat{A}_\nu{}^a - \partial_\nu \hat{A}_\mu{}^a + g f^{abc} \hat{A}_\mu{}^b \hat{A}_\nu{}^c$.[2] [We use a caret to denote operators and the internal symmetry group is taken to be SU(N).]

Our central machinery is the trace anomaly equation for the energy-momentum tensor $\hat{\varphi}_{\mu\nu}$ of the theory with a constant source J coupled to $\hat{\varphi}$. From it, through a Legendre transform, a nonlinear differential equation for the effective potential $V(\varphi)$ is derived [see Eq. (10)], and is solved for small coupling. The solution leads us to conclude that there exists a unique stable vacuum in which $\hat{\varphi}$ condenses with positive sign, relative to the perturbative value, which agrees with that deduced from experiments.[3] It should be emphasized that our discussion does not rely on any assumption of the dominance of certain field configurations nor on the large-N limit.

We then introduce the Wilson loop $\psi(c)$ into the condensed vacuum and derive an exact renormalized equation [see (11)], which states that the area dependence of $\psi(c)$ is determined by how the condensation $\langle\hat{\varphi}\rangle$ changes due to the presence of the loop. The salient feature is that the condensation is broken near the loop.

An elegant derivation of the trace anomaly in QCD[4] has been given by Collins, Duncan, and Joglekar.[5] A new situation arises, however, when one introduces the source J for a "hard" operator $\hat{\varphi}$ and wishes to study the J dependence; one needs to renormalize the theory in such a way that multiple insertions of $\hat{\varphi}$ become finite. This must be fully discussed before we can utilize the method of Ref. 5. Below we shall present the discussion without quarks and later indicate a change to be made when we include them.

Our starting point is the generating functional Z, in an axial gauge, given by

$$Z \equiv \exp(iW) = \int \mathfrak{D}\hat{A}_0 \delta[\eta_\mu \hat{A}_0{}^\mu(x)] \exp\left(i\int d^d x\{-\tfrac{1}{4}(1+J_0)[\hat{G}_0{}^{\mu\nu}(x)]^2 + j_0{}^\mu(x)\hat{A}_{\mu 0}(x)\}\right), \quad (1)$$

where "0" indicates dimensionally regularized bare quantities and d is the dimensionality of space-time. For the purpose of renormalization it is convenient to go to an alternative representation of Z. By making a scale transformation $g_{J0}{}^2 \equiv g_0{}^2/(1+J_0)$, $\hat{A}_{J0}{}^\mu \equiv (1+J_0)^{1/2} A_0{}^\mu$, and $j_{J0}{}^\mu \equiv (1+J_0)^{-1/2} j_0{}^\mu$, Z takes the form (in dimensional regularization the Jacobian is unity),

$$Z = \exp(iW) = \int \mathfrak{D}\hat{A}_{J0} \delta[\eta_\mu \hat{A}_{J0}{}^\mu(x)] \exp\left(i\int d^d x\{-\tfrac{1}{4}[\hat{G}_{J0}{}^{\mu\nu}(x)]^2 + j_{J0}{}^\mu(x)\hat{A}_{J\mu}{}^0(x)\}\right), \quad (2)$$

where $\hat{G}_{J0}{}^{\mu\nu}(x)$ is of the same form as $\hat{G}_0{}^{\mu\nu}(x)$ as a function of $\hat{A}_{J0}{}^{\mu}$ and g_{J0}. Z is made finite by the usual renormalization prescription

$$g_J{}^2 = g_{J0}{}^2 \mu^{\epsilon} Z_3(g_J, \epsilon), \quad \hat{A}_J{}^{\mu} = Z_3(g_J, \epsilon)^{-1/2} \hat{A}_{J0}{}^{\mu}, \quad j_J{}^{\mu} = Z_3(g_J, \epsilon)^{1/2} j_{J0}{}^{\mu}, \quad \epsilon = d-4. \tag{3}$$

The n-fold insertion of the *bare* operator $\hat{\varphi}_0 \equiv \frac{1}{4}\int d^d x [\hat{G}_0{}^{\mu\nu}(x)]^2$ is effected by $(-\partial/\partial J_0)^n W|_{J_0=0}$. Since W can be regarded as a function of $g_{J0}{}^2$ and $j_{J0}{}^{\mu}$, it is easy to establish

$$(-\partial/\partial J_0)^n W|_{J_0=0} = (g_0{}^2)^{-n} D_0{}^n w, \tag{4}$$

where $w \equiv W(J_0=0)$ and $D_0 \equiv g_0{}^2(g_0{}^2 \partial/\partial g_0{}^2 + \frac{1}{2}\int d^d x\, j_0{}^{\mu} \partial/\partial j_0{}^{\mu})$. Obviously, w is a finite function of g^2 and j_{μ} defined by precisely the same form of equations as in (3) with subscript J omitted. Now we can define the *renormalized* n-fold insertion of $\hat{\varphi}$ to be $(g^2)^{-n} D^n w$, where D has the same form as D_0 in terms of g^2 and j_{μ}. Let us see how the bare and the renormalized n-fold insertions are related. All we have to do is to rewrite $(g_0{}^2)^{-n} D_0{}^n$ in terms of $(g^2)^{-n} D^n$ using (3) with $J=0$. The result is

$$(-\partial/\partial J_0) W|_{J_0=0} = z^{-1}(g^2)^{-1} Dw, \quad (-\partial/\partial J_0)^2 W|_{J_0=0} = z^{-2}(g^2)^{-2} D^2 w + z^{-2}(z - 1 - g^2 \partial \ln z/\partial g^2)(g^2)^{-1} Dw, \tag{5}$$

etc., where $z \equiv 1 - g^2 \partial \ln Z_3/\partial g^2$. For general n the structure is

$$(-\partial/\partial J_0)^n W|_{J_0=0} = \sum_{l=1}^{n} C_{nl} (g^2)^{-l} D^l w,$$

where C_{nl} is a function of z and its derivatives with respect to g^2. The gist of all this is that we face an operator mixing with the mixing matrix which is triangular and is of infinite dimension. To handle this situation, we first define the renormalized source J to be such that $(-\partial/\partial J)^n W|_{J=0}$ gives precisely $(g^2)^{-n} D^n w$. Now if we make the relation between J_0 and J *non-linear* and write

$$J_0 = J Z_J(J) = J[Z_J{}^{(0)} + J Z_J{}^{(1)} + \ldots], \tag{6}$$

then the operator mixing above is succintly described. Indeed from (6) we reproduce the form

$$(-\partial/\partial J_0)^n W|_{J_0=0} = \sum_{l=1}^{n} \bar{C}_{nl}[Z_J{}^{(0)}, \; Z_J{}^{(1)}, \ldots](-\partial/\partial J)^l W|_{J=0}.$$

Comparison with previous equations determines $Z_J{}^{(i)}$ in terms of z $[=Z_J{}^{(0)}]$. Moreover, by explicitly solving the defining equation $(-\partial/\partial J)^n W|_{J=0} = (g^2)^{-n} D^n w$, it is not difficult to show that the J dependence of $g_J{}^2$ is quite simple, i.e.,

$$g_J{}^2 = g^2/(1+J). \tag{7}$$

We can now apply the analysis of Ref. 5 to (2) with (6) and (7) in mind. We obtain, in the functional notation of Ref. 5,

$$\int d^4x\, \hat{\theta}_{J\mu}{}^{\mu}(x) = [2\beta(g_J)/g_J][g_J{}^2 \partial/\partial g_J{}^2 + \tfrac{1}{2}\int d^4x\, j_J{}^{\mu}(x)\partial/\partial j_J{}^{\mu}(x)],$$

which, by noting $j_J{}^{\mu}/g_J = j^{\mu}/g$ and (7), can be rewritten as

$$\int d^4x\, \hat{\theta}_{J\mu}{}^{\mu}(x) = [2\beta(g_J)/g_J](1+J)\{\tfrac{1}{4}\int d^4x [\hat{G}_J{}^{\mu\nu}(x)]^2\}, \tag{8}$$

where the operator $\frac{1}{4}\int d^4x [\hat{G}_J{}^{\mu\nu}(x)]^2 \equiv \hat{\varphi}_J$ is defined by $-\partial/\partial J$. This is the desired form of the anomaly equation, which we utilize below. Inclusion of massless quarks does not alter the form of (8). Its effect appears only through a change in the β function.

We are in a position to derive the effective potential for $\varphi \equiv \langle \hat{\varphi}_J \rangle/\Omega$, where angular brackets denote the vacuum expectation value and Ω is the total space-time volume. The operators on both sides of (8) are well-defined except for the perturbative vacuum matrix elements. Since we are interested in possible nonperturbative effects, we shall define them to be zero by subtraction. From Poincare invariance of the vacuum, we have $\langle \hat{\theta}_{J\mu}{}^{\mu}(x)\rangle = 4\langle \hat{\theta}_{J00}(x)\rangle$ so that in fact (8) is an equation for the energy density $\epsilon(J)$ in the presence of the source defined by $\epsilon(J)\Omega = \langle \hat{\theta}_{J00}(x)\rangle \Omega = -W$, i.e.,

$$\epsilon(J) = [\beta(g_J)/2g_J](1+J)\varphi, \quad \partial\epsilon/\partial J = \varphi. \tag{9}$$

The effective potential $V(\varphi)$ is then constructed by $V(\varphi) = \epsilon(J) - J\partial\epsilon(J)/\partial J$.[6] At present there exist no systematic ways of calculating $\epsilon(J)$ or $V(\varphi)$ diagramatically but we do not need them; by substituting

$J=-dV/d\varphi \equiv -V'$ into the definition of $V(\varphi)$ above, we obtain an exact nonlinear differential equation for $V(\varphi)$. For a range of g and J such that g_J is small, it can be solved. With $\beta(g_J)=b_0 g_J^3 + b_1 g_J^5 + \ldots$, the equation becomes

$$V(\varphi)=\tfrac{1}{2}[b_0 g^2 + b_1 g^4/(1-V') + \ldots]\varphi + V'\varphi. \quad (10)$$

In what follows we assume the number of massless quarks to be not too large so that b_0 and b_1 are both negative. We first study (10) for $\varphi \geq 0$. If we keep the $b_0 g^2$ term only, the solution is $V(\varphi)=\varphi - \frac{1}{2}b_0 g^2\varphi \times [\ln(g^2\varphi/\mu^4)+C]$ with the stationary value $\varphi_s=(\mu^4/g^2)\exp(2/b_0 g^2 - C - 1)$. From the definition of one insertion at $J=0$, $\varphi=-g^2\partial V/\partial g^2$, C can be seen to be independent of g.[7] If we retain $b_1 g^4$ as well, the solution becomes $V(\varphi)=\varphi-\frac{1}{2}b_0 g^2\varphi f(\ln(g^2\varphi/\mu^4))$, where the function $f(x)$ is given implicitly by $x=2\int^f dy\{1-y-[(1+y)^2+2b_1/b_0^2]^{1/2}\}^{-1}$. The stationary value is

$$\varphi_s=[\mu^4/(g^2+b_1 g^4/b_0)][-(b_0+b_1 g^2)/g^2]^{-2b_1/b_0^2}\exp(2/b_0 g^2 - C - 1).$$

These solutions are sketched in Fig. 1. It clearly shows that there exists a single stable vacuum whose energy is lower than the perturbative one. Notice that the negative character of b_0 is essential. For $\varphi<0$, it is easy to see from (9) at $J=0$ that the energy becomes higher than that of the perturbative vacuum. The fact that $\varphi=0$ cannot be a stationary point can be shown without assuming small coupling; near $\varphi=0$, $V(\varphi)=-\frac{1}{2}b_0 g^2\varphi \times \ln(g^2\varphi/\mu^4)$ is the exact solution for any value of g.

We now introduce the Wilson loop $\hat{\psi}(c)=(1/N)P \times \mathrm{Tr}\exp[ig\int \hat{A}_\mu(x)dx^\mu]$ in the condensed vacuum. From now on we set $J=0$ throughout. It has been shown[8] recently that $\psi(c)=\langle\hat{\psi}(c)\rangle$ can be made finite by the usual renormalization procedure provided that the self-energy of the test particle is isolated. It is assumed that this has been done. Then the renormalized $\psi(c)$ satisfies $[\mu\partial/\partial\mu + \beta(g)\partial/\partial g]\psi(c)=0$. For simplicity we take a circular loop of radius r, in which case $\psi(c)$ is a function of g and μr. Recalling that $g^2\partial/\partial g^2$ effects an insertion of $\hat{\varphi}[\equiv\int d^4x\hat{\varphi}(x)=\frac{1}{4}\int d^4x\hat{G}_{\mu\nu}{}^2(x)]$, after Wick rotation the above equation can be transformed into

$$[\partial/\partial\sigma + M^2(\sigma)]\psi(c)=0, \quad (11)$$

where σ is the area of the loop and

$$M^2(\sigma)=[\beta(g)/g\sigma]\Delta\Phi, \quad \Delta\Phi\equiv\int d^4x[\varphi_c(x)-\varphi]. \quad (12)$$

Here $\varphi_c(x)$ (φ) is the vacuum expectation value of $\hat{\varphi}(x)$ in the presence (absence) of the loop, i.e., $\varphi_c(x)=\langle\hat{\varphi}(x)\hat{\psi}(c)\rangle/\langle\hat{\psi}(c)\rangle$. It is understood that both $\varphi_c(x)$ and φ have been made finite by subtracting the perturbative value of φ. Diagrammatically M^2 is represented by the exchange of gluons between $\hat{\varphi}$ and the loop, which shifts the expectation value of $\hat{\varphi}$ from its vacuum value. The nontrivial contribution to (11) starts from order g^4. In the large-N limit $\varphi_c(x)=O(N^2)$, $\varphi=O(N^2)$, and $\varphi_c(x)-\varphi=O(1)$. Finiteness of $\psi(c)$ requires $\Delta\Phi$ to be convergent so that $\varphi_c(x)$ approaches the vacuum value φ at infinity faster than $\varphi_c(x)-\varphi\sim[1/(x_\mu^2)^{1/2}]^4$. If $\Delta\Phi$ goes like σ for large σ, then the "area law" for $\psi(c)$ is obtained.

The sign of M^2 deserves a separate discussion. An important property of the Wilson loop is that it tends to reduce the condensation, that is $\Delta\Phi<0$. More precisely external color electric field "expels" the condensation near the source and renders $\varphi_c(x)$ negative. We discuss these phenomena for a small loop.

First we apply the operator product expansion to $\psi(c)$.[9] After simple lowest-order calculations, we obtain, up to logarithms in μr, $\psi(c)\sim 1-g^2(\varphi$

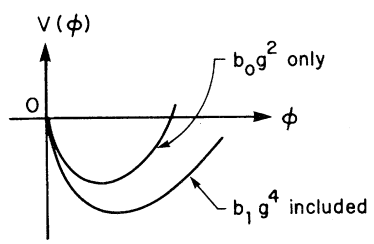

FIG. 1. The effective potential obtained in the text. (Our approximation is valid except for the region where $J=-V'\simeq -1$.)

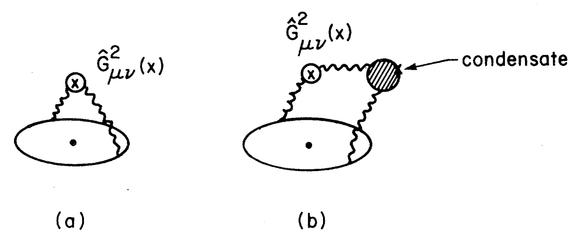

FIG. 2. Lowest-order diagrams contributing to $\varphi_c(x)-\varphi$.

$\times\sigma^2/12N$ + const). Then from (11) and (12), we get $\Delta\Phi = \sigma^2\varphi/6b_0N < 0$.

Next we directly calculate the behavior of $\varphi_c(x) - \varphi$ near the loop. Take a loop in 1–2 plane with its center at the origin and let $x_\mu = (0, 0, 0, h)$. For small h the dominant contribution comes from the diagrams depicted in Figs. 2(a) and 2(b) and is given, up to logarithmic corrections, by

$$\varphi_c(x) - \varphi = -g^2(N^2 - 1/N\pi^4)[\sigma^2/(r^2+h^2)^4] - (g^2/6\pi^2N)[\sigma^2\varphi/(r^2+h^2)^2] < 0.$$

We see that for small r and h the perturbative effect dominates. Since φ is negligible for small coupling, $\varphi_c(x)$ itself is negative.

Equation (11) looks similar to those obtained in Ref. 10. However, our equation differs from theirs in the following respects: Our equation is written in terms of renormalized quantities only and is a first-order differential equation with respect to a global deformation. Note also that the string tension is proportional to the *difference* of $\langle \hat{G}_{\mu\nu}{}^2 \rangle$ inside and outside of the loop. We expect that this will lead to the correct sign of M^2.

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[a]On leave of absence from the Research Institute for Fundamental Physics, Kyoto University, Kyoto, Japan.

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[6]Although we subtracted source-dependent infinities in defining the effective potential, it can still be used to find the stable vacuum, for which the source vanishes; values of V and its derivatives at stationary points have their usual physical significance. Note that in dimensional regularization, the subtracted term actually vanishes.

[7]The above form of the potential was obtained by Fukuda (Ref. 2), assuming the existence of a nontrivial solution to the renormalization-group equation for $\epsilon(J)$. Here, such an assumption is unnecessary.

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