Nonlinear Models in $2+\epsilon$ Dimensions

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A generalization of the nonlinear σ model is considered. The field takes values in a compact manifold M and the coupling is determined by a Riemannian metric on M. The model is renormalizable in $2+\epsilon$ dimensions, the renormalization group acting on the infinite-dimensional space of Riemannian metrics. Topological properties of the β function and solutions of the fixed-point equation $R_{ij}-\alpha g_{ij}=\nabla_i v_j+\nabla_j v_i$, $\alpha=\pm1$ or 0, are discussed.

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Several years ago Polyakov[1] studied the renormalization of the $O(N)$-invariant nonlinear σ model in $2+\epsilon$ dimensions in the low-temperature regime dominated by small fluctuations around ordered states. He found an infrared-unstable fixed point at a temperature of order ϵ. The unstable renormalization-group trajectory gives a model critical system in its scaling limit or, equivalently, a Euclidean quantum field theory.[2] In two dimensions the model is asymptotically free.

I describe here[3] a more general model to which Polyakov's approach is appropriate: a field $\varphi(x)$ taking values in a compact manifold M, governed by the action

$$S(\varphi)=\Lambda^{\epsilon}\int dx\,\tfrac{1}{2}T^{-1}g_{ij}(\varphi(x))\partial_\mu\varphi^i(x)\partial_\mu\varphi^j(x), \qquad (1)$$

where Λ^{-1} is the short-distance cutoff. The dimensionless coupling $T^{-1}g_{ij}$ is a Riemannian metric on M.[4] The standard nonlinear σ models have M a homogeneous space and g_{ij} an invariant metric.

Correlation functions are generated by the partition function

$$Z(h)=\int\prod_x d\varphi(x)\exp[-S(\varphi)+H(\varphi)],$$

where the *a priori* measure $d\varphi(x)$ is the metric volume element on M and $H(\varphi)=\Lambda^{2+\epsilon}\int dx[h(x)](\varphi(x))$, h being an external field, each $h(x)$ a function on M. The k-fold correlation function takes values in the unit measures on M^k:

$$\langle\varphi(x_1)\circ\cdots\varphi(x_k)\rangle =Z^{-1}(0)\frac{\partial}{\partial h(x_1)}\cdots\frac{\partial}{\partial h(x_k)}Z(h)\Big|_{h=0}. \qquad (2)$$

The double expansion in T and ϵ is constructed as a renormalizable perturbation series.[5] Only fields close to the constants play a role; $Z(h)=\int dm\,Z(m,h)$, where $Z(m,h)$ is the sum over small fluctuations around the constant $\varphi(x)=m$. A choice of coordinates around each point m in M gives a linear representation for the fluctuations: The linear field $\sigma^i(x)$ is $\varphi(x)$ in coordinates around m. The sum over fluctuations becomes

$$Z(m,h)=\int d\tilde\sigma\exp[-\tilde S(m,\sigma)+\tilde H(m,\sigma)], \qquad (3a)$$

$$\tilde d\sigma=\prod_x d\sigma(x)\exp[\Lambda^{2+\epsilon}\int dx\ln\det J(m,\sigma(x))], \qquad (3b)$$

$$\tilde S(m,\sigma)=\int dx\,\tfrac{1}{2}T^{-1}\tilde g_{ij}(m,\sigma(x))\partial_\mu\sigma^i(x)\partial_\mu\sigma^i(x), \qquad (3c)$$

$$\tilde H(m,\sigma)=\int dx\,\tilde h(x,m,\sigma(x)), \qquad (3d)$$

where $\tilde g_{ij}(m,\sigma(x))$ and $\tilde h(x,m,\sigma(x))$ are the metric and external field in coordinates around m and $\det J_j{}^i(m,\sigma(x))$ is the Jacobian of the coordinate map from $\sigma(x)$ to $\varphi(x)$. Propagators and vertices come from expansion in powers of σ. Normal coordinates yield

$$J_j{}^i(m,\sigma(x))=\delta_j{}^i+\tfrac{1}{6}\sigma^k(x)\sigma^l(x)R^i{}_{klj}(m)+\dots. \qquad (4a)$$

$$\tilde g(m,\sigma(x))=g_{ij}(m)+\tfrac{1}{3}\sigma^k(x)\sigma^l(x)R_{iklj}(m)+\dots. \qquad (4b)$$

$$\tilde h(x,m,\sigma(x))=\sum_{n=0}^{\infty}(1/n!)\sigma^{k_1}(x)\cdots\sigma^{k_n}(x)\nabla_{k_1}\cdots\nabla_{k_n}[h(x)](m). \qquad (4c)$$

To each constant m corresponds a perturbation series whose vertices are in the most general form required by power counting, so that it is *prima facie* renormalizable. But the existence of an underlying nonlinear theory means that the vertices for one constant m determine those for all nearby m' by translation of coordinates and shift of origin. To renormalize the nonlinear theory the renormalized vertices must be made to satisfy an equivalent renormalized invariance. That this can be done is shown in Ref. 3.

Renormalized as dictated by power counting, at a scale set by μ,

$$g_{ij}=(\Lambda/\mu)^{-\epsilon}g_{ij}{}^{b}(\epsilon,\Lambda/\mu,g^{R}), \tag{5a}$$

$$h(x)=(\Lambda/\mu)^{-2-\epsilon}Z_1(\epsilon,\Lambda/\mu,g^{R})h^{R}(x)+h_1(\epsilon,g), \tag{5b}$$

where $g_{ij}{}^R$ and $h^R(x)$ are the renormalized coupling and external field, Z_1 is a linear operator on functions on M, and h_1 serves to remove quadratic divergences. In the following only renormalized quantities are discussed; the superscripts R are suppressed.

The partition function satisfies the renormalization-group equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\tilde{\gamma}(g)h(x)\frac{\partial}{\partial h(x)}\right)Z(h)=0. \tag{6}$$

The β function $\beta(g)$ is a vector field on the space of metrics and $\tilde{\gamma}(g)h(x)$ is a linear vector field on functions.

The order parameter $\Phi(x)$ dual to $h(x)$ takes values in the nonnegative unit measures on M. The free energy

$$\Gamma(\Phi)=\max_h[-\ln Z(h)+\mu^{2+\epsilon}\int dx\{h(x),\Phi(x)\}] \tag{7}$$

satisfies the renormalization-group equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\gamma^*(g)\Phi(x)\frac{\partial}{\partial\Phi(x)}\right)\Gamma(\Phi)=0, \tag{8}$$

where $\tilde{\gamma}(g)=-(2+\epsilon)+\gamma(g)$.

To two loops,[6] with use of dimensional regularization and renormalizing by minimal subtraction,

$$\beta_{ij}(T^{-1}g)=-\epsilon T^{-1}g_{ij}+R_{ij}+\tfrac{1}{2}T(R_{ikln}R_{jkln})+O(T^2), \tag{9a}$$

$$\gamma(T^{-1}g)=-\tfrac{1}{2}T\nabla_k\nabla_k+O(T^3). \tag{9b}$$

$R_{ij}=R^k{}_{ikj}$ is the Ricci tensor and T has been replaced by $2\pi T$.

The renormalization group has meaning only as it acts on the equivalence classes of metric couplings and external fields under reparametrizations (diffeomorphisms) of M. The partition function $Z(h)$ sees no change when both g_{ij} and h are subjected to the same reparametrization; thus no normalization condition can distinguish among members of the same equivalence class. The construction and renormalization of the perturbation series respect this covariance.

The diffeomorphism classes of metrics make up an infinite-dimensional manifold (singular at metrics with symmetry),[7] over which the external fields form a vector bundle. The renormalization group has its fixed points where $h(x)$ vanishes and $\beta(g)$ is an infinitesmal reparametrization: $\beta_{ij}(g)=\nabla_i v_j+\nabla_j v_i$ for v a vector field on M.

The coefficients β and γ are natural functions of the metric: When g_{ij} is transformed by a reparametrization of M, $\beta_{ij}(g)$ and $\gamma(g)$ undergo the same transformation. In particular, if g is unaffected, then so are β and γ. Thus the renormalization group preserves internal symmetry.

Since a homogeneous space has the same geometry at every point, the couplings of any standard model comprise a finite-dimensional submanifold of the metrics at one point in M. Group-theoretic formulas for renormalization-group coefficients are given in Ref. 3.

Global topological information on the β function for small T is available when M has dimension 2 and also when M is homogeneous. In both cases the β function is a gradient through two loops.[3]

The fixed points correspond to solutions of

$$R_{ij}-\alpha g_{ij}=\nabla_i v_j+\nabla_j v_i,\quad \alpha=\pm 1 \text{ or } 0. \tag{10}$$

Writing the coupling in the form $T^{-1}(g_{ij}+k_{ij})$, with T and k_{ij} small, and keeping only terms of topological significance:

$$\beta(T)=\begin{cases}\epsilon T-\alpha T^2, & \alpha=\pm 1\\ \epsilon T-T^3, & \alpha=0,\ R_{ijkl}\neq 0\\ \epsilon T, & \alpha=0,\ R_{ijkl}=0,\end{cases} \tag{11a}$$

$$\beta(k)_{ij}=\tfrac{1}{2}T\Delta_\beta(k)_{ij},$$

$$\Delta_\beta=-\nabla_1\nabla_1+\text{first order terms}, \tag{11b}$$

$$\tilde{\gamma}=-(2+\epsilon)+\tfrac{1}{2}T\Delta_\gamma,\quad \Delta_\gamma=-\nabla_i\nabla_i-2v^i\nabla_i. \tag{11c}$$

The only meaningful k_{ij} directions are those transverse to the reparametrizations and to the T direction. Δ_β is an elliptic operator with positive leading part, and so the number of unstable or marginal k directions is always finite. The flat metrics ($R_{ijkl}=0$) have trivial perturbation theories; in the following they are excluded from the case $\alpha=0$.

When $\alpha=1$ or 0, there is a nontrivial fixed point for $\epsilon>0$ at $T\approx\epsilon$ or $T\approx\epsilon^{1/2}$, infrared unstable in at least the T direction. When α is -1, there is a fixed point for $\epsilon<0$ at $T\approx-\epsilon$, infrared stable in the T direction. In all three cases, there are also trivial fixed points at $T=0$. No other kind of fixed point at nondegenerate coupling is possible because when the two-loop term in the β function vanishes, i.e., $R_{ikln}R_{jkln}=\nabla_i w_j+\nabla_j w_i$, then $\int dm\, R_{ikln}R_{ikln}=0$, and so $R_{ijkl}=0$.

In two dimensions the trivial and nontrivial fixed points merge at $T=0$, asymptotically free in the small when $\alpha=1,0$ and in the large when $\alpha=-1$. When $\alpha=0$, $\beta(T)$ vanishes to second order in T, and so the approach to freedom is extraordinarily slow.

All known solutions of (10) are actually Einstein metrics ($v^i=0$). For $\alpha=1$, there is available only one example which is not locally homogeneous.[8] Among the homogeneous spaces those admitting just one invariant metric are necessarily Einstein,[9] but others with less symmetry are known.[10] Some have instability in k directions, and so provide model multicritical points.[3] The only known Ricci-flat spaces ($\alpha=0$) are the Kahler manifolds of Yau.[11] Einstein metrics with $\alpha=-1$ are known in two varieties: the locally symmetric spaces of noncompact type and the Kahler metrics of Yau.[11]

For $\epsilon>0$, $\alpha=0$ or 1, the long-distance physics is qualitatively familiar. Below the critical temperature, long-distance behavior is governed by the trivial fixed point at $T=0$, so that there is a degenerate set of pure equilibrium states, labeled by the points in M. At $T=0$ the free energy $\Gamma(\Phi)$ is minimized by the point measures $\Phi_m(x)=\delta_m$. As T increases, the set of minima is still M, but the minimizing order parameters have diffused outward; to lowest order $\Phi_m=\exp(s\Delta_\gamma{}^*)(\delta_m)$, $s=\frac{1}{2}\ln(1-T/T_c)$. At $T=T_c$ the degeneracy of equilibria disappears, the Φ_m having converged to the unique measure annihilated by γ^*. To lowest order the anomalous dimensions of $\Phi(x)$ are determined by the eigenvalues of Δ_γ. Long-distance properties for $T>T_c$ are not accessible to perturbation theory, but the system presumably remains disordered.

A solution of (10) with v^i not a gradient would show some novel features: Approaching the critical surface, the order parameter would drift as it diffused (because of the term $-2v^i\nabla_i$ in Δ_γ) and the anomalous dimensions could be complex.

The $\alpha=-1$ fixed points are analogous to φ^4 fixed points near four dimensions, the ϵ expansion probing dimensions below 2. The scaling limit in two dimensions is trivial, and so it would seem more interesting to attempt an interpretation of the $T=0$ fixed points as the long-distance terminals of trajectories originating on a critical surface at nonzero T. Infrared asymptotic freedom implies a correlation function $\langle\varphi(x)\varphi(0)\rangle$ decaying as $(\ln|x|)^{-\Delta\gamma}$ for large $|x|$. But high-temperature series for lattice versions of the nonlinear models always show finite correlation lengths, and so there must be an intervening phase transition. The locally symmetric $\alpha=-1$ spaces all have nontrivial, non-Abelian fundamental groups, allowing topologically stable vortex-like field configurations. Phase transitions due to dissociation of multivortex bound systems might be expected.[12] Other of the $\alpha=-1$ manifolds, being simply connected, call for different mechanisms.

Construction of a nonstandard model requires the bare *a priori* measure $d\varphi(x)$ which avoids nonspontaneous long-range ordering. For asymptotically small T it can be calculated from the renormalization-group equation for the bare external field. It depends on the method of short-distance regularization and differs from the metric volume element whenever h_1 in Eq. (5b) is nonzero. The difference is of order T, so in two dimensions the critical *a priori* measure is exactly the metric volume element. But an infinite number of relevant couplings (the external fields) must be fixed in order to bring the *a priori* measure to its critical value. In this sense the nonstandard models are unnatural. The standard models have enough internal symmetry to determine the *a priori* measure uniquely, and so for them these issues do not arise.

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[a]Present address.

[1]A. M. Polyakov, Phys. Lett. 59B, 79 (1975).

[2]See K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).

[3]Details and additional material are given in the author's thesis, University of California, Berkeley, 1980 (unpublished).

[4]The geometric character of the coupling of the nonlinear σ model was noted in K. Meetz, J. Math. Phys. (N.Y.) 10, 589 (1969).

[5]E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976).

[6]One-loop results were given in G. Ecker and J. Honerkamp, Nucl. Phys. B35, 481 (1971).

[7]D. G. Ebin, in *Symposium in Pure Mathematics—San Diego—1966*, edited by S. S. Chern and S. Smale (American Mathematical Society, R.I., 1968).

[8]D. N. Page, Phys. Lett. 79B, 235 (1978).

[9]J. A. Wolf, Acta. Math. 120, 59 (1968).

[10]See G. R. Jensen, J. Differential Geometry 8, 599 (1973), and references therein.
[11]S. T. Yau, Proc. Natl. Acad. Sci. U.S.A. 74, 1798 (1977).
[12]Cf. J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).

From Charge-Conjugation Asymmetries to the Trilinear Gluon Coupling

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In quark-quark scattering where a hard gluon is radiated, a charge-conjugation asymmetry is examined and found to be large. The perturbative quantum-chromodynamic prediction for this asymmetry is compared with that for an Abelian theory and an appreciable difference is found. The specific role of the trilinear coupling is significant. A program is set forth for three-jet experiments in hadron-hadron collisions in which the non-Abelian issue can be studied.

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The presently accepted theory of strong interactions, quantum chromodynamics[1] (QCD), is a non-Abelian, color SU(3), gauge theory with an octet of vector gluons as the carriers of the strong force. The experimental testimony[2] on behalf of this theory includes the inferential existence of the quark and gluon, their spin assignments of $\frac{1}{2}$ and 1, respectively, and evidence for the coupling qqg. However, the question of whether the gluon is indeed the color gauge particle with the predicted self-couplings remains unanswered.

There seems to be no easy path leading to the trilinear and quadrilinear couplings of the gluons. The self-coupling effects are generally either very small and/or indirect.[3] The scaling violations in gluon jets may require rather higher energies for a definitive test.

In this Letter, we wish to stimulate interest in a new, direct way of discerning the non-Abelian effects, which shifts the emphasis towards proton machines. This involves charge-conjugation asymmetries in three-jet production from hadron-hadron collisions, and rests on the possibility that jets can be correlated with their parent partons. At large momentum transfers, we appeal to asymptotic freedom and use perturbation theory. As an example, we calculate the asymmetry between quark and antiquark jets for a given gluon jet, and find that there is a significant difference between the Abelian and non-Abelian cases. Both of the color effects, the trilinear coupling and the non-Abelian qqg coupling, are important determinants of this difference.

We first consider the reaction $q\bar{q}\to Q\bar{Q}g$; a reaction which is the basis of the three-jet asymmetry in the hadron collisions of interest. The two quarks, q and Q, are taken to be massless and to have different flavor. The quark charge-conjugation asymmetry is defined in terms of the differential cross section $d\sigma(Q)$ as

$$A=\frac{d\sigma(Q)-d\sigma(\bar{Q})}{d\sigma(Q)+d\sigma(\bar{Q})}.$$

For the calculation of each $d\sigma$, the diagrams of Fig. 1 are needed. Averaging and summing over spin and color, the traces can be checked against expressions to be found in Ref. 4. Classifying the contributions with respect to Q-$\bar{Q}$ interchange, the trigluon coupling contributes only to the symmetric terms (denominator), while the antisymmetric terms (numerator) arise because of initial and final gluon bremsstrahlung interference.

Not only is the asymmetry A nonzero in lowest order, it is generally quite large. In the q-$\bar{q}$ c.m. frame, a benchmark is defined to be the configuration where all the particles in $q\bar{q}\to Q\bar{Q}g$ are

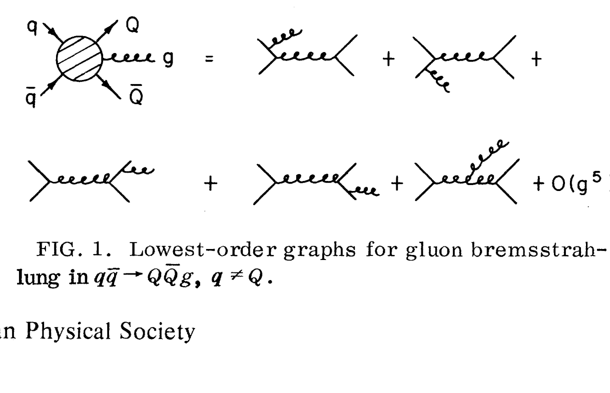

FIG. 1. Lowest-order graphs for gluon bremsstrahlung in $q\bar{q}\to Q\bar{Q}g$, $q\neq Q$.

