# Self-Healing of Confined Plasmas with Finite Pressure

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At finite ratios of the kinetic plasma pressure to the magnetic pressure, the magnetic confinement configurations of axisymmetric plasma columns tend to acquire characteristics that hinder the onset of instabilities driven by the combined effects of magnetic curvature and pressure gradient. A simple analytical dispersion relation that contains the main physical factors affecting an important class of these modes is given.

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The purpose of this Letter is to discuss the maximum value of the ratio $\beta$ of the plasma kinetic pressure to the pressure associated with the confining magnetic field that can be reached without inducing loss of confinement. This ratio is important in order to assess the main characteristics, such as transport properties and rates of radiation emission, of a given magnetic confinement configuration, as well as the type of fusion reactor that can be developed out of it. Here we limit most of our attention to the ideal magnetohydrodynamic (MHD) approximation and notice that when $\beta$ increases toward finite values, instabilities driven by the combined effects of the magnetic field curvature and the pressure gradient can be expected to develop.[1] An essential feature of the ideal MHD model is the strong interaction between the developing plasma instability and its magnetic confinement configuration. As $\beta$ is increased, the configuration of the plasma isobaric surfaces also changes, and, since the plasma motion is tied to that of the magnetic lines, this will have a dual effect[2]: Not only will it increase the instability driving pressure gradient, but it also will enhance the stabilizing magnetic tension in that region of unfavorable magnetic curvature where the relevant modes develop.

In the earliest stability analysis of these modes, the considered equilibrium models neglected the modification of the confinement configuration that takes place as $\beta$ evolves and becomes finite. In particular, only linear terms in the plasma pressure gradient were retained in the relevant normal mode equation, and rather pessimistic upper limits on $\beta$ for stable configurations were obtained. However, as we shall show in this Letter, important nonlinear terms must also be retained; as was first demonstrated for simplified equilibrium configurations, these terms can lead to production of a "second stability region."[3,4] This circumstance is illustrated by the following dispersion relation that we have derived from a consistent description of the ideal MHD equilibrium configuration in the vicinity of the magnetic axis:

$$-\omega^2 \simeq \frac{3}{5}\left(\frac{v_{A0}G^2}{q_0R_0}\right)^2\left(\frac{\hat{s}}{G^2}-\frac{5}{6}\right)^2\left(\frac{\hat{s}}{G^2}-\frac{5}{32}\right). \quad (1)$$

Here we have employed familiar notations except for the dimensionless parameters

$$\hat{s}(\psi) \equiv d\ln q(\psi)/d\ln r(\psi),$$

$$G(\psi) \equiv -8\pi R_0 q_0 B_0^{-1} r(\psi) dp(\psi)/d\psi$$

that measure the magnetic shear and the plasma pressure gradient. For $G^2 < 1.2\hat{s}$, corresponding to the first stability region in the $(G, \hat{s})$ plane, the pressure gradient is not strong enough to overcome the stable shear-Alfvên oscillations. For $G^2 > 6.4\hat{s}$ the tendency of the plasma to expand is hindered by the increased magnetic tension, and the considered confinement configurations are again stable.

In the limit of high toroidal number, incompressible plasma displacements have been shown to satisfy the following normal-mode equation[1,5]:

$$\Gamma^2(1+\Sigma^2)\Upsilon(\theta) = B_p^2R^2\hat{J}^{-1}\frac{d}{d\theta}\left[\hat{J}^{-1}B_p^{-2}R^{-2}(1+\Sigma^2)\frac{d\Upsilon(\theta)}{d\theta}\right] + GB_pR_0q_0r^{-1}B_0^{-1}[-R\kappa_n + R\kappa_g\Sigma]\Upsilon(\theta). \quad (2)$$

We have adopted as coordinates the toroidal angle $\zeta$, the polar angle $\theta$ around the magnetic axis, and the radial flux function $r(\psi) = (2q_0\psi/B_0)^{1/2}$, where $B_0$ and $q_0$ are the magnetic field and the inverse rotational transform at the magnetic axis. The eigenfunction $\Upsilon$ is related[4,6] to the component of the displacement normal to the magnetic surface; the dimensionless Jacobian $\hat{J}$ is defined by $dV = \hat{J}R_0 r\, dr\, d\theta\, d\zeta$; the inverse rotational transform is $q = (2\pi)^{-1}\oint q_l\, d\theta$ and the quantity $\Sigma$ is

$$\Sigma = R^2 B_p^2 q_0 (BB_0 r)^{-1}[\partial/\partial r + |\nabla r|^{-2}\nabla\theta \cdot \nabla r \partial/\partial\theta]\int^\theta q_l\, d\theta'. \quad (3)$$

In addition, $\Gamma = i\omega R_0 q_0/v_{A0}$ and the normal and geodesic components of the magnetic curvature are represented by $\kappa_n$ and $\kappa_g$, respectively. The Alfvén velocity $v_{A0}$ is referred to the field at the magnetic axis: $v_{A0}^2 = B_0^2/(4\pi\rho)$. All other symbols have their usual meanings. In Eq. (2) the variable $\theta$ should be regarded as a transformed one, ranging from $-\infty$ to $+\infty$ (see Refs. 7–9).

The linear stability of MHD modes described by Eq. (2) in finite-$\beta$ configurations can be investigated by studying the properties of that equation in the vicinity of the magnetic axis, where analytical solutions of the finite-$\beta$ MHD equilibrium are known. We prescribe the poloidal flux function $\psi$ to vanish at the magnetic axis; thus both $r(\psi)$ and $G(\psi)$ tend to zero as $\psi^{1/2}$ and it is convenient to expand Eq. (2) in powers of $G$. The equilibrium configuration will be assumed to be marginally stable against localized interchanges to lowest order ($q_0 = 1 +$ ellipticity corrections),[10] so that the terms associated with these modes in the MHD energy principle do not overwhelm the ballooning-mode terms we are interested in, as we approach the magnetic axis. The coefficients in Eq. (2) are periodic functions of $\theta$ except for the secular term $\hat{s}\theta$ in $\Sigma$. Around the magnetic axis, the shear $\hat{s}$ tends to zero like $\psi$, and we can introduce the finite parameter $g \equiv \lim_{\psi\to 0} G^2/4s$. Since the variable $\theta$ extends to infinity, solving our differential equation when $\hat{s}\to 0$ requires a multiple scale analysis: We split the dependence on $\theta$ into a "periodic" $\theta$ and a "stretched" variable $l \equiv \hat{s}\theta$.

To write down explicit representations for the coefficient functions in Eq. (2) we use the following equilibrium solution, suitably expanded in powers of $G$:

$$d(\psi,\theta) = r(\psi)\{(1+\lambda\cos 2\theta)^{-1/2} - 2^{-3}(1+\lambda\cos 2\theta)^{-2}[1+(1+\lambda)/(4\beta_{p0})]\cos\theta\, G(\psi) + F_2(\theta)G^2(\psi) + F_3(\theta)G^3(\psi) + \dots\}, \quad (4)$$

where $d(\psi,\theta)$ represents the polar distance from the magnetic axis on a meridian plane, $\beta_{p0} \equiv -2\pi R_0^2 q_0 B_0^{-1} dp(0)/d\psi$ is a local measure of the poloidal $\beta$, $\lambda$ ($|\lambda| < 1$) determines the ellipticity of the flux surfaces close to the magnetic axis, and $F_2$ and $F_3$ are rather involved functions of the poloidal variable that are given by Coppi, Ferreira, and Ramos[11]; triangularity and other free shape parameters are set equal to zero.

This equilibrium can be used to obtain perturbative solution of Eq. (2). The eigensolution is accordingly expanded in powers of $G$: $\Upsilon(l,\theta) = \Upsilon_0(l,\theta) + G\Upsilon_1(l,\theta) + \dots$, and we require that it be a periodic function of $\theta$. The terms of order unity in (2) yield the equation

$$\frac{\partial}{\partial\theta}\left[A_0(l,\theta)\frac{\partial\Upsilon_0(l,\theta)}{\partial\theta}\right] = \Gamma_0^2 C_0(l,\theta)\Upsilon_0(l,\theta), \quad (5)$$

where $A_0 = 1 + l^2(1+\lambda^2) + 2\lambda(l^2\cos 2\theta - l\sin 2\theta)$ and $C_0 = (1+\lambda\cos 2\theta)^{-2}A_0$ are positive definite functions. Therefore the only acceptable solution is $\Gamma_0 = 0$, $\Upsilon_0 = \Upsilon_0(l)$. Subsequently we can solve for the periodic dependences in $\Upsilon_1$, $\Upsilon_2$, and $\Upsilon_3$. To order $G^3$, the right-hand side of Eq. (2) vanishes after integration over the periodic variable $\theta$. Since $\oint(1+\Sigma^2)d\theta$ is $\sim 1$, it follows that the leading term in the expansion of $\Gamma^2$ is of order $G^4$, and we can write

$$\Gamma^2 = \hat{\Gamma}^2 g^{-2}(G/2)^4 + O(G^5) = \hat{\Gamma}^2\hat{s}^2 + O(G^5).$$

By averaging the terms of order $G^4$ over one period, we obtain the eigenvalue equation for $\hat{\Gamma}^2$ and $\Upsilon_0(l)$:

$$\hat{\Gamma}^2(1+l^2)\Upsilon_0(l) = \frac{d}{dl}\left[(1+l^2)\frac{d\Upsilon_0(l)}{dl}\right] + [A(g)(1+l^2)^{-1} - B(g)]\Upsilon_0(l), \quad (6)$$

where

$$A(g) = 4g - (\tfrac{3}{2})g^2 + \lambda[(\tfrac{11}{2})g^2 - 7g], \quad (7)$$

$$B(g) = (\tfrac{3}{8})g^2 - \lambda(7g^2 - g)/4. \quad (8)$$

These expressions (7), (8) are only correct to first order in $\lambda$ and in the limit $\beta_{p0} \gg 1$. In addition to the parameter $g$, the equilibrium function $F_3(\theta)$ in Eq. (4) involves another parameter [proportional to $d^2p(0)/d\psi^2$] that has nevertheless disappeared from Eq. (6). Thus, to lowest order in $G$ and for $\beta_{p0} \gg 1$, the stability limits and the growth rates depend only on the parameter $g$ and the ellipticity $\lambda$.

Equation (6) can be regarded as a "distilled" version of the original ballooning mode equation

(2), which contains the essential information about how the equilibrium configuration affects the onset of the instability. Thanks to its simplicity, it also provides a useful basis to evaluate the mode growth rate. First of all there is a simple analytical solution at marginal stability ($\hat{\Gamma}^2=0$), $\Upsilon_0(l)=(1+l^2)^{-\gamma}$, where the exponent $\gamma$ and the marginally stable value of $g$ are the solutions of the algebraic equations

$$\gamma=(\tfrac{1}{4})\{1+[1+4B(g)]^{1/2}\}=[A(g)/4]^{1/2}. \qquad (9)$$

These admit two solutions,

$$g_{\rm I}=0.30+0.54\lambda,\ \gamma_{\rm I}=(\tfrac{4}{15})^{1/2}+0.02\lambda;$$

$$g_{\rm II}=1.60\pm 3.46\lambda,\ \gamma_{\rm II}=0.80\pm 0.02\lambda,$$

confirming the existence of two points of marginal stability. Notice that positive values of $\lambda$ (i.e., vertically elongated magnetic surfaces) shift both marginal points towards higher values of $g$, showing a tendency to make the first stability region wider and the second one narrower. Within the range of values of $\lambda$ for which our linear perturbation computations are reliable, the discriminant $1+4B(g)$ remains positive. Therefore the considered configurations are stable against localized interchanges to order $G^4$, an expected result since we are dealing with high poloidal $\beta$.

In order to estimate the positive growth rates which are found for $g_{\rm I}<g<g_{\rm II}$, we can make use of the quadratic form

$$\hat{\Gamma}^2\int_{-\infty}^{\infty}dl\,(1+l^2)\,|\Upsilon_0|^2=-\int_{-\infty}^{\infty}dl\,(1+l^2)\,|d\Upsilon_0/dl|^2+g\int_{-\infty}^{\infty}dl\,\{(1+l^2)^{-1}[4-(\tfrac{3}{2})g]-(\tfrac{3}{8})g\}\,|\Upsilon_0|^2. \qquad (10)$$

For the sake of simplicity we have omitted the contributions of ellipticity. This equation clearly shows the stabilizing $g^2$ terms which imply that the growth rate $\Gamma^2$ cannot increase indefinitely with the pressure gradient parameter $G$, and will eventually fall back to zero at a second point of marginal stability.[4] Given the value of the exponent $\gamma_{\rm I}$ characterizing the eigenfunction $\Upsilon_0(g=g_{\rm I};\,l)$, we see that the growth rate must be depressed around the first marginal point $g_{\rm I}$. As a matter of fact, the approximate expression

$$\hat{\Gamma}^2\simeq\tfrac{3}{20}g^2(g^{-1}-\tfrac{10}{3})^2(g^{-1}-\tfrac{5}{8})$$

fits the $\hat{\Gamma}^2(g)$ curve obtained by numerical integration of Eq. (6), within errors of less than 5% relative to the maximum value of $\hat{\Gamma}^2$, for $g_{\rm I}<g<g_{\rm II}$. From this we derive the approximate dispersion relation, Eq. (1). The latter can be generalized to include the effects of small ellipticity as follows:

$$-\omega^2\simeq(v_{A0}G^2/q_0R_0)^2(\tfrac{3}{5})(1+3.2\lambda)[\hat{s}/G^2-(\tfrac{5}{6})(1-1.8\lambda)]^2[\hat{s}/G^2-(\tfrac{5}{32})(1-2.2\lambda)]. \qquad (1')$$

Considering the dispersion relation, Eq. (1), near the points of marginal stability, we have to take into account the effects of finite ion gyroradius.[12] Thus, in Eq. (1), $\omega^2$ should be replaced by $\omega(\omega-\omega_{di})$, where $\omega_{di}=cn^0(en)^{-1}dp_i/d\psi$, $p_i$ is the ion pressure, $n$ is the particle density, and $n^0$ the toroidal mode number. Then instability is found only for $\Gamma^2>(\omega_{di}q_0R_0)^2(2v_{A0})^{-2}$, corresponding to

$$\Gamma^2>GR_0(\rho_i k_\theta)^2(dp_i/d\psi)[16r_{pi}(dp/d\psi)]^{-1},$$

where $k_\theta$ is the poloidal wave number, $\rho_i$ is the ion gyroradius, and $r_{pi}=(-d\ln p_i/dr)^{-1}$ is the ion pressure gradient scale distance.

A second aspect is that for a small positive $\hat{\Gamma}^2$, the relevant asymptotic solution is $\Upsilon_0\sim l^{-1}\times\exp(-\hat{\Gamma}l)$, and this is acceptable to the extent that

$$(k_r\rho_i)^2\sim(k_\theta\rho_i l)^2\sim(k_\theta\rho_i)^2/\hat{\Gamma}^2\sim(k_\theta\rho_i)^2\hat{s}^2/\Gamma^2<1.$$

Finally we notice that the boundary between the unstable region and the second stability region may be representative of the pressure radial distribution for a quasiaxisymmetric magnetic configuration that results from the excitation of high-$n^0$ ballooning modes when reaching the first instability boundary. In fact, when this boundary is reached, during a heating process, the plasma pressure tends to overcome the stiffness of the magnetic field lines and leads to a growth of the local value of $G$ by steepening the pressure gradient. At the same time, the characteristic length of variation of the confining field along the poloidal direction tends to shorten and the features of the configurations that lie in the second stability region tend to be produced. It is possible that the combination of the physical effects we have described is responsible for the lack of experimental observation of ballooning modes in the experiments that appear to have exceeded the first stability boundary.

In this connection we notice that a recent numerical analysis[13] has shown that an axisymmetric configuration in which ideal MHD ballooning modes become unstable tends to evolve into a new, nonaxisymmetric equilibrium configuration characterized by relatively sharp current density profiles. These profiles are in fact similar to those found for axisymmetric equilibrium configurations driven to relatively high values of $\beta$ through magnetic-flux conservation. Then a potential limitation to high-$\beta$ confinement can arise from those modes which produce magnetic reconnection and are driven unstable by the plasma current-density gradient. However we observe that, on the basis of the linear theories developed so far, these modes are considerably more difficult to excite in collisionless regimes than in regimes where the effects of collisional electrical resistivity are important. Thus the collisionless plasmas that have to be produced before reaching thermonuclear conditions should sustain without disruption current-density gradients and values of the rotational transform on axis $\iota_0 = 1/q_0$ well above the levels achieved in present-day experiments, and therefore favor the realization of finite-$\beta$ confinement as well.

We also note that the stabilizing effects described in the present Letter depend on the constraint that the plasma motion is tied to that of the magnetic field lines. On the other hand, the same constraint does not allow the plasma stability to benefit from the magnetic well that is dug at finite values of $\beta$, because this well is carried along with the plasma itself. Conversely, the presence of a magnetic well is expected to exert a favorable influence on modes that can be found outside the ideal MHD approximation.

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