neglected contributions to the internal ionization due to direct collisions of the β particle with the K electrons and contributions due to internal Compton scattering of the IB on the K electrons.

The contribution of the direct-collision mechansim to autoionization in β decay has been estimated by Feinberg[9] as $B_K P_K / E_\beta$, which gives a lower limit in the case of ^{204}Tl of $\sim 0.1 P_K$. Although this is a lower limit only, a comparison of the experimental values of P_K with existing theoretical calculations[11] which neglect the collision effect indicates that this contribution is unlikely to be appreciably higher.

An order-of-magnitude estimate of the contribution of the internal Compton scattering of the IB can be obtained by use of Compton-scattering cross sections on the K electrons. Assuming hydrogenic wave functions for the K electrons one gets for the contribution of this effect a value of $\sim 0.1 P_K$.

It should also be noted that the KUB calculations underestimate the intensity of the IB in the case of ^{204}Tl by about 20%. The agreement between the measured intensity of the IB in ^{204}Tl and the predictions of the theory has been shown[16] to be significantly improved when detour transitions are taken into account.

Thus the total contribution of mechanisms other than shakeoff and emission of IB in transitions via intermediate electron states, neglecting interference effects, may amount to $\sim 40\%$ of the value of $P_{\mathrm{IB},K}$ estimated on the assumption that this is the only effect present.

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[a] Present address: The American Oncologic Hospital, Fox Chase Center, Philadelphia, Pa. 19111.

[1] A. Zide and H. Lancman, Phys. Rev. C 19, 1053 (1979).
[2] H. Leutz and K. Ziegler, Nucl. Phys. 55, 648 (1964).
[3] J. M. Howard, E. J. Seykora, and A. W. Waltner, Phys. Rev. A 4, 1740 (1971).
[4] B. P. Pathak, Phys. Rev. C 10, 1599 (1974).
[5] K. Narasimhamurty and S. Jnanananda, Proc. Phys. Soc. London 90, 109 (1967).
[6] R. A. Ricci, Physica (Utrecht) 24, 297 (1958).
[7] J. K. Knipp and G. E. Uhlenbeck, Physica (Utrecht) 3, 425 (1936).
[8] F. Bloch, Phys. Rev. 50, 272 (1936).
[9] E. L. Feinberg, J. Phys. (U.S.S.R.) 4, 423 (1941).
[10] P. Stephas and B. Crasemann, Phys. Rev. 164, 1509 (1967).
[11] C. S. Wang Chang and D. L. Falkoff, Phys. Rev. 76, 365 (1949).
[12] R. R. Lewis and G. Ford, Phys. Rev. 107, 756 (1957).
[13] A. J. Mord, in *Proceedings of the International Conference on Inner Shell Ionization Phenomena and Future Applications, Atlanta, Georgia, 1972*, edited by R. W. Fink, J. T. Manson, I. M. Palms, and R. V. Rao, CONF-720404 (U. S. Atomic Energy Commission, Oak Ridge, Tenn., 1973), Vol. 3, p. 2079.
[14] L. Madansky *et al.*, Phys. Rev. 84, 596 (1951).
[15] S. B. Nilsson, Ark. Fys. 10, 457 (1956).
[16] R. J. D. Beattie and J. Byrne, Nucl. Phys. A161, 650 (1971).

Sequential-Scattering Model for Relativistic Heavy-Ion Collisions

Rudi Malfliet
Kernfysisch Versneller Instituut, 9700av Groningen, The Netherlands
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A microscopic model, constructed within the framework of the Boltzmann equation, is formulated and used to calculate particle spectra from high-energy nucleus-nucleus collisions. Formation of composite particles is treated according to Hagedorn's statistical-thermodynamical approach of strong interactions. Inclusive proton, deuteron, triton, and ^{3}He cross sections from 400-MeV/nucleon ^{20}Ne + ^{208}Pb are calculated and found in good agreement with experiment.

Recent experiments at the Bevelac on medium- and high-energy heavy-ion collisions have produced impressive inclusive measurements of double-differential cross sections.[1] They were obtained for protons, light ions, and also pions over a large momentum and angular range and supplement the first results[2] for this type of heavy-ion reaction. Attempts for a theoretical description[3] of the measured proton spectra have been based mainly on two different assumptions.

The first one implies a rapid equilibration of the degrees of freedom and replaces the dynamical evolution of the collision process by simple kinematics and conservation laws. The models[4,5] based on this simple idea of thermal equilibrium have had some success when compared with the data, although the drastic assumption involved poses many questions. The second type of approach starts from the full recognition of the initial collision process as a microscopic nonequilibrium phenomenon which eventually shows certain equilibrium features. Assuming a certain degree of diluteness of the colliding system, the governing equation is the Boltzmann equation. If one uses statistical techniques to solve this equation one performs a so-called Monte Carlo (cascade) calculation. Doing this, one has been able to reproduce the gross features of the proton spectra although some differences between various calculations remain.[6-8]

In addition to the emission of collision constituents (protons, neutrons), the production of composites in these reactions presents an interesting probe to our understanding of the reaction process, especially since the major part of their spectrum is nonevaporative. It has been analyzed in terms of a coalescence mechanism[9] where nucleons with appropriate relative momenta collide and form composites, the underlying main potential acting as a catalyst. Alternatively, one pictures in the thermal equilibrium approach the formation of composite nuclei as originating from an expanding collection of strongly interacting particles raised to a high temperature.[10] This system evolves from a high density towards a low-density free expansion. Both models have a parameter, the critical coalescence momentum and the value for the freezeout density, respectively. Up to now, the thermal-equilibrium prescription is the only one available which gives a qualitative and in some cases also a quantitative description of the proton as well as the composites spectra in a unified way.[11] Therefore it would be valuable to see what new insight can be obtained by starting from the nonequilibrium *Ansatz*. In this Letter, I will present the essential ingredients and some results of a theoretical model which follows this line of thought.

The Boltzmann equation describes the evolution in phase space of the one-particle distribution function $N(\vec{r},\vec{v},t)$ and, in case of no external force with only two-body collisions, has the following form:

$$(E\partial/\partial t+\vec{p}\cdot\nabla)N(\vec{r},\vec{v},t)=\iiint d\omega_2 d\omega_1' d\omega_2' [N(\vec{r},\vec{v}_1',t)N(\vec{r},\vec{v}_2',t)W(\vec{v}_1',\vec{v}_2'|\vec{v}_2,\vec{v}) - N(\vec{r},\vec{v},t)N(\vec{r},\vec{v}_2,t)W(\vec{v},\vec{v}_2|\vec{v}_1',\vec{v}_2')], \tag{1}$$

where W denotes the transition probability (which contains the nucleon-nucleon cross section) and $d\omega=d^3p/E$ (momentum $\vec{p}$, energy E). Equation (1) is the covariant form of the Boltzmann equation for the invariant distribution function N.[12] This nonlinear equation can be solved by statistical methods, which, although correct, are fairly nontransparent and might suffer from insufficient statistics. In view of this I have developed another solution scheme which, based on certain (plausible) assumptions, simplifies (1) a great deal while gaining in physical insight.

In the cascade resulting from the collision of projectile and target nucleus one can distinguish two mechanisms. There is the direct production of cascade nucleons and the interaction amongst them (rescattering) which will affect their momentum distribution. Since we have already a substantial smearing out of the momentum spectrum due to the Fermi motion in the colliding nuclei, I will neglect the effect of the rescattering on the direct cascade and treat this process separately. This amounts to a linearization of Eq. (1) where I allow only for collisions between a cascade nucleon (distribution function N) and a so-called spectator nucleon (distribution function N_s) either from projectile or target. This linearized Boltzmann equation is then solved by an iteration scheme ($N=\sum_n N_n$) based on the order n of scatterings. The first term on the right-hand side of the (linearized) Eq. (1) is taken as the scattering kernel which, when omitted, defines the zeroth-order solution. This iteration procedure will converge rapidly since the average number of collisions is small (long mean free path of a nucleon in a nucleus together with rapid depletion of the initial densities). Finally I assume that, on the average, the production of cascade nucleons and the corresponding depletion of the spectator part takes place sequentially by increasing order of scattering. In this way the time dependence can be eliminated explicitly by introducing time-integrated distribution functions $\overline{N}$ and $\overline{N}_n$. The time evolution is then replaced in some sense by the order of scatterings.

The Boltzmann Eq. (1) is thus reduced to the following (sequential-scattering model) integral equation for the time-integrated distribution function $\bar{N}(\vec{r},\vec{v})=\sum_n \bar{N}_n(\vec{r},\vec{v})$, where the index n indicates order of scattering:

$$\bar{N}_n(\vec{r},\vec{v})=\int_{-\infty}^{0} d\tau\, C_n(\vec{r},\vec{v},\tau)\exp\left[-\int_{\tau}^{0} d\tau' \int d\omega'\, d\omega''\, W(\vec{v},\vec{v}''|\vec{v}')\bar{N}_s^{(n)}(\vec{r}+\vec{v}\tau',\vec{v}'')\right], \tag{2}$$

$$C_n(\vec{r},\vec{v},\tau)=\int d\omega''\, d\omega'\, \bar{N}_{n-1}(\vec{r}+\vec{v}\tau,\vec{v}')\bar{N}_s^{(n)}(\vec{r}+\vec{v}\tau,\vec{v}'')W(\vec{v}',\vec{v}''|\vec{v}).$$

Conservation of momentum has been used to eliminate one of the velocity variables in the transition probability W. I have used the notation $\bar{N}_s^{(n)}$ for the average depleted spectator distribution after n collisions. The zeroth-order $\bar{N}_0$ solution of (2) corresponds to the penetration probability for the projectile through the target (or vice versa). Then $\bar{N}_n$ is the probable number of nucleons produced after n collisions and their subsistence in $(\vec{r},\vec{v})$ space is governed by the exponential quenching factor taking into account the loss because of higher-order scatterings on their way out. Evaluating $\bar{N}(\vec{r},\vec{v})$ asymptotically then allows one to calculate cross sections for producing nucleons. It is worthwhile to point out that there exists a close connection between the iterative solution of Eq. (2) and the Glauber multiple-scattering approach.[13]

One important aspect we have not yet taken into account is the occurrence of collisions between n-times-scattered nucleons. These will affect the final spectra and give rise to a faster equilibration. Also the formation of light composite particles has not been dealt with. However, these same interactions between outgoing nucleons (after n scatterings) can produce, in the spirit of the coalescence model, the composites. In principle the collisions between the "moving-out" nucleons can be included in Eq. (2) but the simultaneous production and absorption of composite particles is much harder to handle. A much simpler technique to take these interactions into account is based on the idea that their effect can be shifted to the density in phase space: A gas of interacting particles can be described as a gas of noninteracting particles including all particles, bound states of particles, resonances, and a newly created particles allowed by conservation laws.[14] This approach has been further developed by Hagedorn[15] to treat multiparticle production in high-energy p-p collisions. If we have a volume element dV, sufficiently small such that velocity and temperature T can be considered as constant, the interacting hadronic matter inside can be described as a statistical equilibrium of all kinds of hadrons, including resonances. This equilibrium is a local one and needs no time to be established nor a large number of collisions to produce it.[15] The momentum distribution of particles of kind i [mass m_i, energy $E_i=(m_i^2+p^2)^{1/2}$, multiplicity g_i, chemical potential μ_i] is then given, in its rest frame, by the familiar expression

$$f_i(\vec{p},T)=\frac{g_i dV}{2\pi^2}\frac{p^2 dp}{\exp[(E_i-\mu_i)/T]\pm 1}. \tag{3}$$

One can now combine both the thermodynamical description (3) of the interactions mentioned before and the solution of the approximate Boltzmann Eq. (2).

This is done as follows: After each iteration n, there is a number of nucleons described by their velocity distribution function $\bar{N}_n(\vec{r},\vec{v})$. Since the thermodynamical spectrum (3) does not depend on $\vec{r}$ explicitly but only on the fact that at any $\vec{r}$ there is a specific momentum spectrum, I can carry out the integration over $\vec{r}$ (adding all dV contributions to the final spectrum) independently. In this way I obtain from $\bar{N}_n(\vec{r},\vec{v})$ a weight function $F_n(\lambda)$ which counts all contributions coming from all volume elements to a given longitudinal momentum (represented by its rapidity λ). For each λ there is also a corresponding average transversal energy (also calculated from the velocity distribution function) from which the the temperature can be calculated. The total number of nucleons at the nth collision together with baryon-number conservation and the assumption of chemical equilibrium ($\mu_A=A\mu_1$) is then used to determine μ_1, the chemical potential of a nucleon. The integrated colume V is not much different from the natural interaction volume, i.e., the volume of the nucleus, and that leaves essentially no free parameters. The final momentum spec- spectrum $W_i(\vec{p})$ of particles of kind i (also nucleons) is then obtained, for a given impact, from an expression like

$$W_i(\vec{p})=\sum_n \int_{-1}^{+1} d\lambda\, F_n(\lambda) L_\lambda(\vec{p}'\to\vec{p}) f_i((\vec{p}',T_n(\lambda)), \tag{4}$$

where $L_\lambda(\vec{p}'\to\vec{p})$ denotes the Lorentz transformation relating the rest frame $(\vec{p}')$ to the moving frame $(\vec{p})$. An expression of the form (4) has been written down first by Hagedorn,[15] although

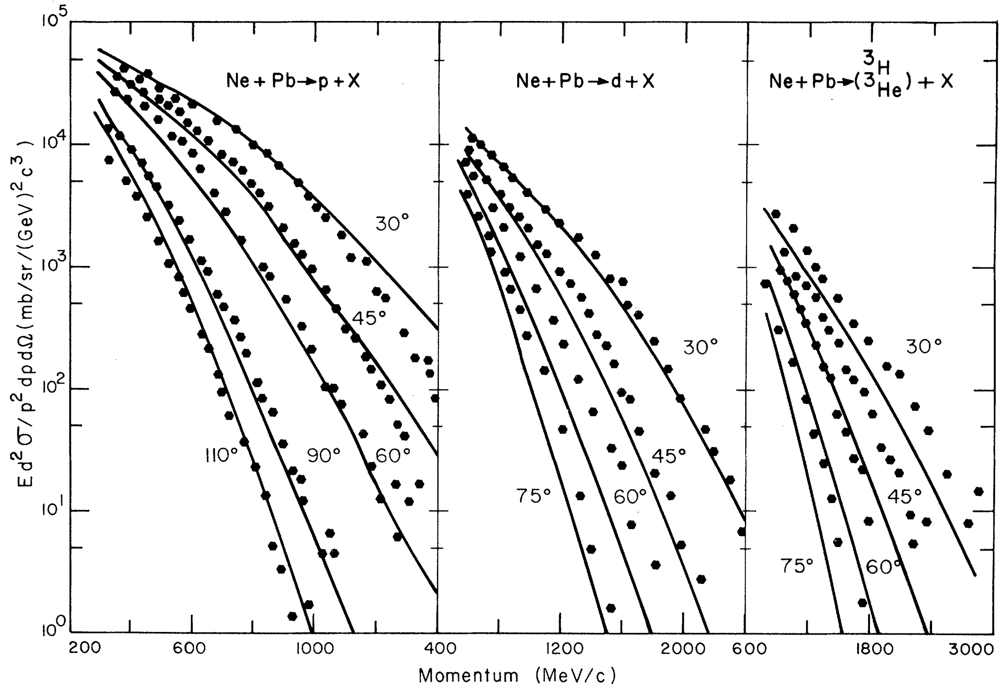

FIG. 1. Single-particle inclusive cross sections for production of protons, deuterons, and triton plus ^{3}He combined, in 400-MeV/nucleon ^{20}Ne + Pb. The solid line is based on calculations presented in this Letter.

in his case he was not able to compute the weight function $F_n(\lambda)$.

Based on the model outlined above I have performed some calculations on the particle spectra from high-energy heavy-ion reactions. The sequential-scattering Eq. (2) was solved numerically, including relativistic kinematics, experimental total nucleon-nucleon cross sections and correct treatment of fermi motion in projectile and target. Pion degrees of freedom were not included. The initial nuclear densities were of Saxon-Woods form and their depletion during the collision process was taken into account. The index i was restricted to nucleons and all known bound states of nucleons up to $A = 5$ only. After each scattering event (index n) the appropriate weight functions $F_n(\lambda)$, temperatures $T(\lambda)$, and chemical potentials μ_i were determined. Finally, expression (4) was evaluated for each impact parameter to obtain the final cross sections.

In Fig. 1, I display the comparison with experimental data for one typical case, i.e., 400-MeV/nucleon Ne + Pb.[1] The theoretical calculation shows good agreement with experimental data without invoking radical assumptions or parameters. It is a microscopic calculation which does not assume thermal equilibrium and treats both nucleons and composites within the same nonequilibrium framework.

It remains an intriguing question how this and other (thermal) models will perform on more exclusive data.

[1]S. Nagamiya, I. Tanihata, S. Schnetzer, L. Anderson, W. Brückner, O. Chamberlain, G. Shapiro, and H. Steiner, Lawrence Berkeley Laboratory Report No. 6770 (unpublished).

[2]J. Gosset, H. H. Gutbrod, W. G. Meyer, A. M. Poskanzer, A. Sandoval, R. Stock, and G. D. Westfall, Phys. Rev. C 16, 629 (1977).

[3]J. R. Nix, Los Alamos Scientific Laboratory Report No. 77-2952, 1977 (to be published).

[4]G. D. Westfall, J. Gosset, P. J. Johansen, A. M. Poskanzer, W. G. Meyer, H. H. Gutbrod, A. Sandoval, and R. Stock, Phys. Rev. Lett. 37, 1202 (1978).

[5]W. D. Myers, Nucl. Phys. A296, 177 (1978).

[6]A. A. Amsden, J. N. Ginocchio, F. H. Harlow, J. R. Nix, M. Danos, E. C. Halbert, and R. K. Smith, Phys. Rev. Lett. 38, 1055 (1977).

[7]Z. Fraenkel and Y. Yariv, unpublished.

[8]J. D. Stevenson, Phys. Rev. Lett. 41, 1702 (1978).

[9]H. H. Gutbrod, A. Sandoval, P. J. Johansen, A. M. Poskanzer, J. Gosset, W. G. Meyer, G. D. Westfall, and R. Stock, Phys. Rev. Lett. 37, 667 (1976).

[10]A. Mekjian, Phys. Rev. Lett. 38, 640 (1977).

[11]J. Gosset, J. I. Kapusta, and G. D. Westfall, Phys. Rev. C 18, 844 (1978).

[12]S. R. de Groot, C. G. van Weert, W. Th. Hermens, and W. A. van Leeuwen, Physica (Utrecht) 40, 257 (1968).
[13]R. A. Malfliet and Y. J. Karant, Nuclear Science Annual Report, 1975 (unpublished), and Lawrence Berkeley Laboratory Report No. LBL-5075, 270 (1977).
[14]E. Beth and G. E. Uhlenbeck, Physica (Utrecht) 4, 915 (1937).
[15]R. Hagedorn and J. Ranft, Suppl. Nuovo Cimento 6, 169 (1968).