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Singularities and Scaling Functions at the Potts-Model Multicritical Point

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Differential renormalization equation for the q-state Potts model are proposed, and the critical behavior of the model near $q=q_c$ discussed. The equations give rise to critical and tricritical fixed points which merge at $q=q_c$ when a dilution field becomes marginal, to an essential singularity in the latent heat as a function of $q-q_c$, in accordance with the exact result of Baxter, and, for $q=q_c$, to a logarithm correction to the power-law behavior of the free energy as a function of $T-T_c$.

A large amount of information is available regarding the behavior of the q-state Potts model in two dimensions for q near its critical value $q_c=4$. From Baxter's[1] exact calculations it is known that this model exhibits a second-order phase transition for $q<q_c$ and a first-order transition for $q>q_c$. Furthermore, the latent heat associated with this first-order transition exhibits an essential singularity in $q-q_c$ as q approaches q_c from above. Recently, Nienhuis and co-workers[2,3] developed a renormalization-group approach in which disordered spin cells of the Potts model were identified with vacancies. In this enlarged space of dilute Potts Hamiltonians the topology of the phase diagram responsible for the behavior near q_c corresponded to a smooth curve consisting of a line of critical fixed points which change to a line of tricritical fixed points at the critical value of q_c.

In order to gain insight into the nature of the region near q_c we carried out a Migdal-Kadanoff[4] approximate renormalization transformation on the dilute Potts model.[5] Motivated by these results, we developed from general analytic considerations a set of differential renormalization relations for the relevant scaling fields[6] when q is near q_c. Here we present these equations and discuss their physical consequences. These relations imply that the critical exponent associated with the dilution field ψ becomes marginal when the fixed points merge at $q=q_c$. As a consequence of this property, we find the following: (1) As $q\to q_c^{\,+}$, the latent heat for the pure Potts limit ($\psi<0$) has an essential singularity in $(q-q_c)^{1/2}$ as was first calculated by Baxter[1]; (2) in the dilute limit ($\psi>0$) of the Potts model the latent heat vanishes with an essential singularity in the dilution field ψ; and (3) for $q=q_c$ the free energy has a logarithmic correction to the power-law behavior as a function of the thermal field $\varphi\propto T-T_c$.

The introduction of a dilution field[2] allows the sudden switch of the pure Potts model from a second-order to a first-order transition in the φ-q plane to be viewed analytically in the extended ψ-φ-q space. The variable q maps onto itself and is treated simply as a parameter. The differential renormalization-group equations for φ and ψ involve parameters which can be obtained for the two-dimensional case by fitting the coefficient in these equations to den Nijs's conjecture[7] near $q=q_c$ and Baxter's result[1] for the essential singularity in the latent heat.

We believe that this approach is quite general and can be applied to related problems, e.g., the Baxter eight-vertex model and the Ashkin-Teller model,[8] which exhibit variable critical exponents, giving a specific realization to the concept that these models all lie in the same universality class.[9,10] An essential idea is to add a new field variable which does not appear in the original model, e.g., the dilution field ψ in the case of the pure Potts model, in order to obtain analytic renormalization transformations. A well-known example of this is Kosterlitz's[11] differential renormalization equation for the planar model in which a dilution field variable was introduced. In fact, our equation, Eq. (1), corresponds to a reduced form of one of Kosterlitz's equations. One would expect that similar relations occur for all models with critical lines belonging to the same universality class.

In differential form the renormalization-group equations for ψ and φ are

$$d\psi(x)/dx = a[\psi^2(x) + \mu^2], \tag{1}$$

$$d\varphi(x)/dx = [y + b\psi(x)]\varphi(x), \tag{2}$$

where a, b, and y are constants to be determined, $\mu = (q - q_c)^{1/2}$, and dx is an infinitesimal fractional change in scale. The singular part of the free energy is given by

$$f_s(\varphi, \psi; \mu) = e^{-xd} f_s(\varphi(x), \psi(x); \mu), \tag{3}$$

where $\varphi(x)$ and $\psi(x)$ are taken outside the critical region, and the correlation length ξ scales as $f_s^{-1/d}$. To obtain the singularities of f_s we will need to fix the value of $\psi(x)$ in suitable domains. We discuss separately the choices $\psi(x) = 1$, 0, and -1, with $\varphi(x) = C$ defining corresponding scaling functions $f_s(C, \psi(x); \mu)$ along the flows determined by Eqs. (1) and (2). There are, of course, corrections to these equations involving other powers of φ and ψ, but we have shown[5] that these will not alter the singular behavior of the free energy as $\varphi \to 0$.

For $q < q_c$, $\mu = i(q_c - q)^{1/2}$, Eqs. (1) and (2) give critical and tricritical fixed points at $\psi_c = -|\mu|$, $\varphi_c = 0$ and $\psi_t = |\mu|$, $\varphi_c = 0$, respectively. The corresponding thermal exponents associated with φ are $y_c = y - b|\mu|$ and $y_t = y + b|\mu|$, while the exponents associated with the dilution field ψ are $y_t' = -y_c' = y' = 2a|\mu|$. We note that the thermal exponents have the extended analytic form of den Nijs's conjecture for $d = 2$ near $q = q_c = 4$ with $y = \frac{3}{2}$ and $b = 3/4\pi$. In the limit $q = q_c$, the critical and tricritical fixed points coalesce as was discovered by Nienhuis and co-workers[2] giving rise to a single fixed point with a marginal exponent $y' = 0$. For $q < q_c$ and $\psi_c < \psi < \psi_t$ there is a crossover behavior with the thermal exponent varying between y_c and y_t.

In order to study the singularities of the free energy we integrate Eqs. (1) and (2) obtaining

$$x = (\mu a)^{-1}\{\tan^{-1}[\psi(x)/\mu] - \tan^{-1}(\psi/\mu)\} \tag{4}$$

and

$$c = \varphi\left(\frac{[\psi(x)]^2 + \mu^2}{\psi^2 + \mu^2}\right)^{b/2a} e^{yx}, \tag{5}$$

where $-\frac{1}{2}\pi \leq \tan^{-1}(\psi/\mu) \leq \frac{1}{2}\pi$.

For $q > q_c$ we must explicitly assume the existence of a latent heat at $\varphi = 0$, by the requirement that f_s have a term proportional to φ with different coefficients depending upon whether $\varphi \to \pm 0$. This behavior depends on the global properties of the renormalization group and in particular on the existence of a discontinuity fixed point. Of course these properties are not described by the local differential relations, Eqs. (1) and (2), which are appropriate near the region where the critical and tricritical lines meet, but the full Migdal-Kadanoff recursion relations which we have derived for the dilute Potts model do contain a discontinuity fixed point and have a behavior of the type we now require. Here we proceed by assuming that $\lim_{c \to \pm 0} f_s(c, 1; \mu)$ contains a term proportional to C but with coefficients which depend on the sign of C:

$$\lim_{C \to \pm 0} f_s(C, 1; \mu) \cong A_\pm C + A. \tag{6}$$

In Eq. (6) $A_\pm$ and A depend on μ and must approach constant values when $\mu \to 0$, because $f_s(C, 1; \mu)$ is regular at $\mu = 0$.[12] Hence, for arbitrary ψ, Eqs. (3), (5), and (6) imply that $\lim_{\varphi \to \pm 0} f_s(\varphi, \psi; \mu)$ contains a term

$$A_\pm \varphi\left(\frac{1 + \mu^2}{\psi^2 + \mu^2}\right)^{b/2a} \exp\left\{\frac{(d - y)}{a\mu}\left[\tan^{-1}\frac{\psi}{\mu} - \tan^{-1}\frac{1}{\mu}\right]\right\} \tag{7}$$

and correspondingly the latent heat $L = \lim_{\varphi \to 0}(\partial f_+/\partial\varphi - \partial f_-/\partial\varphi)$ is given by

$$L = (A_+ - A_-)\left(\frac{1 + \mu^2}{\psi^2 + \mu^2}\right)^{b/2a} \exp\left\{\frac{(d - y)}{a\mu}\left[\tan^{-1}\frac{\psi}{\mu} - \tan^{-1}\frac{1}{\mu}\right]\right\}. \tag{8}$$

If we now consider the limit $\mu \to 0$ of Eq. (8), we find different results depending on whether $\psi < 0$ or $\psi > 0$ corresponding, respectively, to the pure and the dilute Potts domains:

$$\lim_{\mu\to 0} L \cong \frac{(A_+ - A_-)}{|\psi|^{b/a}} \exp\left[-\frac{(d-y)\pi}{a\mu}\right], \quad \psi<0;$$
$$\lim_{\mu\to 0} L = \frac{(A_+ - A_-)}{\psi^{b/a}} \exp\left[-\frac{(d-y)}{a}\left(\frac{1}{\psi}-1\right)\right], \quad \psi>0. \tag{9}$$

Hence for $\psi < 0$ the latent heat vanishes with an essential singularity[13] in μ, in accordance with Baxter's[1] exact result $d=2$, $L \sim \exp[-\pi^2/2(q-q_c)^{1/2}]$. Since $y=\frac{3}{2}$ in this case, we conclude that $a=1/\pi$, where a is the constant introduced in Eq. (1). For $\psi>0$ the latent heat has also an essential singularity as $\psi \to 0$ and vanishes in this limit as we would expect.

If we take the limit $\mu \to 0$ of $f_s(\varphi,\psi;\mu)$ with $|\varphi| > 0$, Eq. (7) remains valid only for $\psi > 0$. In the case $\psi \leq 0$ we find that $C \to \infty$ as $\mu \to 0$, and we would need the appropriate asymptotic behavior of the scaling function $f_s(C,1;\mu)$ Instead we consider the choice $\psi(x) = -1$ in Eqs. (4) and (5) which defines the scaling function $f_s(C,-1;\mu)$. In the limit $\mu \to 0$, $\psi \leq 0$, we have

$$C = \frac{\varphi}{|\psi|^{b/a}} \exp\left[-\frac{y}{a}\left(\frac{1}{|\psi|}-1\right)\right], \tag{10}$$

which is now finite, and

$$f_s(\varphi,\psi;\mu) = \exp\left[\frac{d}{a}\left(\frac{1}{|\psi|}-1\right)\right] f_s(C,-1;\mu). \tag{11}$$

Imposing the condition that $f_s(\varphi,\psi;\mu)$ be regular in the limit $|\psi| \to 0$ now dictates the form of $f_s(C,-1;\mu)$ as $C \to 0$ for $\mu = 0$:

$$f_s(C,-1;\mu) \cong B|C|^{d/y} |\ln|C||^{-s}, \tag{12}$$

where B is a constant and $s = db/ya$. Note that $b/a = (y_t - y_c)/y'$, where $y_c = y - b|\mu|$, $y_t = y + b|\mu|$, and $y' = 2a|\mu|$ are the scaling exponents previously obtained from Eqs. (1) and (2). Substituting Eqs. (10) and (12) in Eq. (11) we obtain to leading order in small φ and ψ

$$f_s(\varphi,\psi;0) \cong B|\varphi|^{d/y} |y/a + \psi \ln|\varphi||^{-s}. \tag{13}$$

For $d=2$ we have $a = 1/\pi$, $b = 3/4\pi$, and $y = \frac{3}{2}$, which implies $s=1$.

Hence for $q = q_c$ the familiar power-law behavior for scaling is modified in the pure Potts domain, $\psi < 0$, by a logarithmic correction due to the existence of a marginal exponent, $y' = 0$, at the multicritical fixed point.[14] At $\psi = 0$ this marginal exponent does not play a role, and the logarithmic correction vanishes. This may account for the fact that the Baxter-Wu[15] model, which belongs to the same universality class as the Potts model, nevertheless does exhibit pure power-law behavior with $y = \frac{3}{2}$.

From Eq. (13) we can now deduce the nontrivial asymptotic behavior of the scaling function $f_s(C,1;\mu)$ when $C \to \infty$, and μ is small:

$$\lim_{C\to\infty} f_s(C,1;\mu) \approx B|C|^{d/y} \left|\ln|C| - \frac{y\pi}{a\mu}\right|^{-s}. \tag{14}$$

The same approach can be applied to obtain the scaling functions and critical behavior in the domain $q \leq q_c$, where $\mu = i(q_c - q)^{1/2}$, and we summarize here only our results. To obtain a scaling function in the crossover region $-|\mu| < \psi < |\mu|$, we set $\psi(x) = 0$ and find from Eqs. (4) and (5)

$$C = \varphi \frac{|1+\psi/|\mu||^{y_c/y'}}{|1-\psi/|\mu||^{y_t/y'}}. \tag{15}$$

Now the condition that $f_s(\varphi,\psi;\mu)$ have no singularities at $\psi = \pm|\mu|$ for $\varphi \neq 0$ implies that near the critical and tricritical fixed points we have

$$f_s(\varphi,\psi;i|\mu|) = D_c(\mu)|\varphi|^{d/y_c}, \quad \psi \sim -|\mu| \tag{16}$$

and

$$f_s(\varphi,\psi;i\mu) = D_t(\mu)|\varphi|^{d/y_t}, \quad \psi \sim +|\mu|, \tag{17}$$

exhibiting the familiar crossover behavior in the critical exponent. The condition that at $\psi = 0$ f_s have a unique limit as $\mu \to 0$ implies

$$D_c(0) = D_t(0) = (a/y)^s B, \tag{18}$$

where B is the constant in the limiting form of the scaling function $f_s(C,-1;0)$, Eq. (12).

For $\psi > |\mu|$ we can obtain the latent heat L from the scaling function $f_s(C,1;\mu)$, Eq. (6):

$$L \sim (A_+ - A_-)\left(\frac{\psi - |\mu|}{1-|\mu|}\right)^{(d-y_t)/y'} \left(\frac{1+|\mu|}{\psi+|\mu|}\right)^{(d-y_c)/y'} \tag{19}$$

which reduces in the limit $|\mu| \to 0$ to Eq. (9).

Finally, for $\psi \lesssim -|\mu|$

$$f_s(\varphi,\psi;i|\mu|) = \frac{D_c|\varphi|^{d/y_c}}{[-\psi+|\mu|]^{ys/y_c}} [2|\mu| + \epsilon]^{ys/y_c}, \tag{20}$$

where

$$\epsilon = \frac{(-|\mu|-\psi)}{(|\mu|-\psi)^{y_t/y_c}} |\varphi|^{y'/y_c} (2|\mu|+\epsilon)^{y_t/y_c}, \tag{21}$$

which in the limit $|\mu| \to 0$ corresponds to Eq. (13) and in the limit $\varphi \to 0$ to Eq. (16).

These results can also be extended to include the effects of symmetry-breaking fields. One finds logarithmic corrections to power-law scaling for the conjugate order parameter at $q = q_c$, and an essential singularity in $q - q_c$ for quantities like the discontinuity in the zero-field magnetization at $T = T_c$, in analogy to Eqs. (13) and (9), respectively.[16]

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[12]This example illustrates the importance of choosing appropriately the value of $\psi(x)$ to deduce the analyticity of the scaling function. For $\psi(x) \leq 0$ we know that the latent heat vanishes as $\mu \to 0$, but in this case we do not know, *a priori*, the dependence on μ of $A_\pm$.

[13]This clarifies the connection between Baxter's result and Kosterlitz's essential singularity in the correlation function of the planar model as a function of $(T - T_c)^{1/2}$.

[14]An alternative approach which leads to the same result, Eq. (13), is to parametrize the flows in terms of $\psi(x) = C$ fixing $\varphi(x)$ outside the critical region, say $\varphi(x) = 1$. The corresponding regularity condition on the scaling function $f_s(1, C; \mu)$ now becomes $\lim_{C\to 0} f_s(1, C; \mu) = \text{const.}$

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Two-Spin Correlation Functions of an Ising Model with Continuous Exponents

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An Ising model on a square lattice is studied, where one row of horizontal bonds has an energy E_1' different from all other horizontal bonds. The correlation of two spins is calculated in this row, resulting in exponents β and η which depend on E_1'. The long-distance behavior of the correlation for fixed $T \neq T_c$ is found to have different forms depending upon the value of E_1'.

Since the discovery by Baxter[1] of a two-dimensional (2D) statistical mechanical model whose specific heat exponent depends on the parameters of the Hamiltonian there has been widespread recognition that many other 2D models such as the massive Thirring model,[2] the Ashkin-Teller model,[3] and the planar rotator[4] [O(2)] model will have correlation functions with continuous critical indices. For the region where the mass gap vanishes these models all bear some relation in leading order to the Gaussian model.[5] However, when there is a mass gap the only exact information known is the correlation length of the Baxter model.[6] It is thus extremely interest-

