Universal Ratios Among Correction-to-Scaling Amplitudes and Effective Critical Exponents

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Writing thermodynamic quantities near a critical point as $f_i \sim A_i|t|^{-\lambda_i}(1+a_i|t|^{\Delta})$, where $t=(T-T_c)/T_c$, we show that to leading order in $\epsilon = 4-d$ one has $a_i/a_j = (\lambda_i - \lambda_i^0)/(\lambda_j - \lambda_j^0)$, where λ_i^0 is the mean-field value of λ_i. This ratio is also equal to $(\lambda_{i,\mathrm{eff}} - \lambda_i)/(\lambda_{j,\mathrm{eff}} - \lambda_j)$, where $\lambda_{i,\mathrm{eff}}$ is derived from a fit of data with $f_i \sim |t|^{-\lambda_{i,\mathrm{eff}}}$. Various experiments are analyzed and compared to these predictions.

Some seven years ago it was realized, both experimentally[1] and theoretically,[2] that analyses of data taken not too close to a critical point should include *"correction-to-scaling" confluent singularity* terms. A thermodynamic quantity f_i should thus be written as

$$f_i = A_i|t|^{-\lambda_i}[1 + a_i|t|^{\Delta} + O(|t|^{2\Delta})], \qquad (1)$$

where $t=(T-T_c)/T_c$. The exponent Δ has since been estimated by various methods,[3-5] and its best values are $\Delta = 0.493$, 0.521, and 0.550 for $n=1$, 2, and 3 component systems.[4] Experimental results, which are fitted with the pure power law $f_i \simeq A_{i,\mathrm{eff}}|t|^{-\lambda_{i,\mathrm{eff}}}$, will thus yield values of $\lambda_{i,\mathrm{eff}}$ given by[6]

$$\lambda_{i,\mathrm{eff}} = \lambda_i - a_i\Delta|\bar{t}_i|^{\Delta} + O(|\bar{t}_i|^{2\Delta}), \qquad (2)$$

where $\bar{t}_i$ is some average temperature.[7] Thus, experimentally determined effective exponents may differ significantly from their predicted universal values λ_i.

Although many theoretical results are now known about *universal relations* among the leading critical amplitudes,[8] not much information is available on *ratios among the correction amplitudes* a_i, which are also predicted to be universal. High-temperature series expansions[5] show that a_ξ^+/a_χ^+ is universal (ξ and χ are the correlation length and the susceptibility, while the plus means $T>T_c$), with the values 0.70 ± 0.03 ($n=1$) and 0.6 ± 0.1 ($n=2$). More recently, Chang and Houghton[9] calculated a_C^+/a_C^- (C is the specific heat, while the minus means $T<T_c$) to order ϵ^2 ($d=4-\epsilon$ is the dimensionality).

In this paper we show that, to leading order in ϵ, one has

$$a_i/a_j = (\lambda_i - \lambda_i^0)/(\lambda_j - \lambda_j^0) + O(\epsilon), \qquad (3)$$

where λ_i^0 is the mean-field value of λ_i. If two effective exponents are measured over a similar temperature range, then Eqs. (2) and (3) also imply that

$$(\lambda_{i,\mathrm{eff}} - \lambda_i)/(\lambda_{j,\mathrm{eff}} - \lambda_j) = a_i/a_j \qquad (4)$$

[if two effective exponents are measured over different temperature ranges, then the left-hand side of Eq. (4) should be multiplied by $|\bar{t}_j/\bar{t}_i|^{\Delta}$]. Such universal ratios should be of great help in the analysis of experimental data involving effective exponents. Equation (4) simply says that a plot of $\lambda_{i,\mathrm{eff}}$ vs $\lambda_{j,\mathrm{eff}}$ should fall on a universal straight line, whose slope is equal to a_i/a_j. It is interesting to note that to $O(\epsilon^0)$, Eqs. (3) and (4) imply that the *effective exponents obey all the thermodynamic scaling relations* (e.g., $\alpha_{\mathrm{eff}} + 2\beta_{\mathrm{eff}} + \gamma_{\mathrm{eff}} = 2$), *but not those of hyperscaling* (e.g., $\alpha_{\mathrm{eff}} + d\nu_{\mathrm{eff}} = 2$ will have a correction of order $\Delta|t|^{\Delta}$, resulting from the fact that the mean-field exponents λ_i^0 do not obey hyperscaling). Thermodynamic scaling relations do indeed seem to be obeyed by many effective exponents.[10] A discussion of experimental ratios is given below.

The result (3) is a direct consequence of the renormalization-group trajectory integral procedure recently developed by Rudnick and Nelson.[11] They considered a Landau-Ginzburg-Wilson–type Hamiltonian, of the form

$$\mathcal{H} = -\int d^3R(\tfrac{1}{2}r|\vec{S}|^2 + \tfrac{1}{2}|\Delta\vec{S}|^2 + u|\vec{S}|^4), \qquad (5)$$

and solved the differential recursion relations explicitly. The parameter u turns out to flow to its fixed-point value, $u^* = 2\pi^2\epsilon/(n+8) + O(\epsilon^2)$, according to $u(l) = u^*/R(l)$, with $R(l) = 1 + We^{-\epsilon l}$, and $W = (u^* - u)/u$. Similarly, the flow of the temperature variable, $t = r + (n+2)u/4\pi^2$, is given by $t(l)$

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$=te^{l/\nu}(Ru/u^*)^{-4\nu_1}$, where $\nu \simeq 1/2+\nu_1\epsilon$. For $T>T_c$, the recursion relations are iterated until $l=l^*$, with $t(l^*)=1$, at which point the various thermodynamic quantities are obtained by perturbation theory. Since all the lengths in the problem have been rescaled by e^{l^*}, the correlation length becomes

$$\xi \propto e^{l^*}=t^{-\nu}(Ru/u^*)^{2\nu_1}, \tag{6}$$

with, to leading order in ϵ,

$$R=1+Wt^{\epsilon/2}. \tag{7}$$

Similarly one finds for the specific heat and for the susceptibility[11]

$$C=(A_+/\alpha)t^{-\alpha}R^{2\alpha_1}+C_B, \quad \chi=\Gamma t^{-\gamma}R^{2\gamma_1}, \tag{8}$$

with $\alpha \simeq \alpha_1\epsilon$, $\gamma \simeq 1+\gamma_1\epsilon$, etc. Note that the *same* factor R enters into *all* these correction powers. It is this term that gives the logarithmic corrections (e.g., $C \propto |\ln t|^{2\alpha_1}$) at $\epsilon \to 0$.[12] For small t, $R^{2\nu_1}$ can be expanded[13] to yield $1+2\nu_1 Wt^{\Delta}$, thus identifying $\Delta=\epsilon/2+O(\epsilon^2)$ and $a_\xi^+=2\nu_1 W+O(\epsilon)$. For $T<T_c$ one determines l^* from[11] $t(l^*)=-\frac{1}{2}$, so that $e^{-\epsilon l^*}\simeq(-2t)^{\epsilon/2}$. Since $2^{\epsilon/2}=1+O(\epsilon)$, the zeroth-order results remain unchanged. The factor $2^{\epsilon/2}$ should however appear, together with other factors, in the order-ϵ corrections to ratios like a_i^+/a_i^-. In summary, our zeroth-order results are

$$\begin{aligned} a_C^+ &= a_C^- = 2\alpha_1 W=\frac{4-n}{n+8}W, \\ a_\chi^+ &= a_\chi^- = 2\gamma_1 W=\frac{n+2}{n+8}W, \\ a_\xi^+ &= a_\xi^- = 2\nu_1 W=\frac{1}{2}\frac{n+2}{n+8}W, \\ a_M &= -2\beta_1 W=\frac{3}{n+8}W, \end{aligned} \tag{9}$$

with W being a nonuniversal parameter.

It is interesting to note that to the leading order we have, e.g., $\gamma_1=(\gamma-\gamma^0)/\epsilon$, $\beta_1=(\beta-\beta^0)/\epsilon$ with $\gamma^0=1$, $\beta^0=\frac{1}{2}$ being the exponents' mean-field values.[14] This is the source of Eq. (3). There is *a priori* no reason to expect that the $O(\epsilon)$ terms in Eq. (3) vanish. In fact Chang and Houghton[9] find nonzero $O(\epsilon)$ corrections.[14] Since extrapolation to $\epsilon=1$ is ambiguous, one can obtain two alternative estimates for a_i/a_j by either (a) use of Eq. (9), or (b) use of Eq. (3) [without the $O(\epsilon)$ corrections] with the actual asymptotic exponents taken from Ref. 4. The difference between the two should be viewed as an estimate of the uncertainties. For instance, Eq. (9) yields $a_\xi^+/a_\chi^+=\frac{1}{2}$, whereas Eq. (3) yields $a_\xi^+/a_\chi^+\simeq 0.55$, somewhat closer to the corresponding series value.[5]

To lowest order in ϵ, Eqs. (3) and (4) immediately give all the thermodynamic scaling relations among effective exponents. However,[10,12]

$$\begin{aligned} 2-\alpha_{\text{eff}}-d\nu_{\text{eff}} &= (2-d/2)a_\xi\Delta|t|^{\Delta}/(\nu-\tfrac{1}{2}) \\ &\simeq W\Delta|t|^{\Delta}+O(\epsilon^2). \end{aligned}$$

The next term in the expansion of $R^{2\nu_1}$ is[13] $\nu_1(2\nu_1-1)W^2|t|^{2\Delta}$, and one can easily find universal relations among these higher-order corrections. Note that our statements about scaling relations among effective exponents still hold.

The same kind of ratios may be derived for any problem which involves ϵ expansions. For example, m-component *dipolar magnets* should have[15]

$$\begin{aligned} a_C^+/a_\chi^+ &= (m^2-2m-4)/(m+2)^2+O(\epsilon) \\ &= \alpha/(\gamma-1)+O(\epsilon), \\ a_M/a_\chi^- &= (4+3m)/(m+2)^2+O(\epsilon) \\ &= -(\beta-\tfrac{1}{2})/(\gamma-1)+O(\epsilon). \end{aligned} \tag{10}$$

Similarly, the corresponding quantities in the *percolation problem* should have,[16] e.g.,

$$-a_M/a_C^- = a_C^+/a_\chi^+ = 1+O(6-\epsilon). \tag{11}$$

We now turn to a review of the experimental situation. Whenever we quote ratios based on the actual asymptotic exponents λ_i, we take these from the perturbation-series calculation.[4] High- and low-T series and extrapolated ϵ-expansion values are not significantly different.[4] Reasonably accurate direct measurements of ratios like a_i/a_j are currently available only for the superfluid transition in ^{4}He,[1,17] where one has $a_C^+/a_C^- = 1.29\pm 0.25$. We predict $1+O(\epsilon)$. (See also Ref. 9.) The same work[17] gives a_C^-/a_{ρ_s} [ρ_s is the superfluid density, which behaves as $|t|^{\zeta}(1+a_{\rho_s}|t|^{\Delta})$, with $\zeta=(d-2)\nu$] in the range -0.13 to -0.21.[17] Since the relation $\zeta=(d-2)\nu$ involves the dimensionality, one does not have $a_{\rho_s}=(2-d)a_\xi$. Instead, Rudnick and Jasnow[18] find that to leading order $\rho_s=M^2$, *including all the corrections*. Thus, $a_{\rho_s}/a_M=2+O(\epsilon)$, and we predict $a_C^-/a_{\rho_s}=(4-n)/6+O(\epsilon)\simeq\frac{1}{3}$ for $n=2$. The real exponents[4] yield $a_C^-/a_{\rho_s}\simeq -0.03$. Clearly, better experimental and theoretical values are needed.

Recent measurements near the liquid-gas critical point of ^{3}He have yielded[19] $a_M/a_\chi^+=0.41\pm 0.2$, from a fit with a single confluent correction term and $|t|\leq 10^{-2}$. Here, the uncertainty includes the

TABLE I. Experimental effective exponents and correction-to-scaling–amplitude ratios.

	β_{eff}	γ_{eff}	a_M/a_χ [f]
^3He [a]	0.358	1.168	0.46
^4He [a]	0.356	1.188	0.59
Xe [a]	0.350	1.211	0.86
CO_2 [a]	0.349	1.199	0.58
H_2O [a]	0.350	1.225	1.60
O_2 [a]	0.353	1.190	0.56
"Avg" [a]	0.355	1.19	0.60
^3He [b]	0.35	1.19	0.50
Ni [c]	0.378	1.34	0.29
EuO [d]	0.368	1.29	0.03
Pd_3Fe [e]	0.444	1.17	0.37

[a] Liquid-gas system ($n = 1$), Ref. 20.
[b] Liquid-gas system ($n = 1$), Ref. 19.
[c] Ferromagnet ($n = 3$), Ref. 21.
[d] Ferromagnet ($n = 3$), Ref. 22.
[e] Ferromagnet ($n = 3$), Ref. 23.
[f] Left-hand side of Eq. (4), with λ_i from Ref. 4.

effects of systematic errors in T_c.[19] We obtain $1+O(\epsilon)$ [Eq. (19), $n=1$] or $0.175/0.24=0.73+O(\epsilon)$ [Eq. (3), exponents from Ref. 4]. The agreement is reasonably good.

There are a number of other experiments which yield *effective exponents*, but by themselves do not have the resolution to yield the amplitudes a_i directly. For these, we compare the left-hand side of Eq. (4) (with λ_i from Ref. 4, see last column in Table I with Eq. (9), or Eq. (3) with the λ_i of Ref. 4. The experimental results for a_M/a_χ are summarized in Table I. The liquid-gas critical points yield, on the average, $a_M/a_\chi \cong 0.6$ (the result for H_2O appears anomalously high), whereas we predict $1+O(\epsilon)$ [Eq. (9)] or 0.73 [from Eq. (3) and the exponents of Ref. 4]. The agreement with our second estimate is remarkably good. The last three rows in the table correspond to Heisenberg systems ($n=3$). For Ni and ordered Pd_3Fe, a_M/a_χ is distinctly different from the liquid-gas case, but in good agreement with our estimate 0.35 based on the exponents of Ref. 4. From the ϵ expansion we get $\frac{3}{5}+O(\epsilon)$. The results for the ferromagnet EuO do not seem to fit very well into the same category as Ni and Pd_3Fe. Perhaps this is attributable to the dipolar interactions which are more important in the case of EuO; Eq. (10) with $m=d=4-\epsilon$ gives $a_M/a_\chi=\frac{4}{9}+O(\epsilon)$, which is smaller than the value $a_M/a_\chi=\frac{1}{2}+O(\epsilon)$ resulting from Eq. (9)! Also, the dipolar exponents λ_i may differ from the short-range ones[4] used here.

For a_ξ^-/a_χ^-, the left-hand side of Eq. (4) (with λ_i from Ref. 4) and experimental data for $\lambda_{i,\mathrm{eff}}$ for the liquid-gas critical point[24] of SF_6 yields 0.67. Our estimates, given above, are 0.5 and 0.55.

In general, our ratios seem to be in reasonable agreement with measured effective exponents. It should be noted that we did not include measurements which were done very close to T_c,[25] since the correction terms there may be too small, and the resulting errors in ratios like a_i/a_j too large.

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Universal Ratios Among Correction-to-Scaling Amplitudes on the Coexistence Curve

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Correction-to-scaling amplitudes are calculated to second order in $\epsilon=4-d$ by renormalized perturbation theory. It is shown that ratio of any two such amplitudes is universal, and that thermodynamic scaling laws among effective exponents break down beyond zeroth order in ϵ. Values found are $a_\xi^+/a_\chi^+=0.65$ ($n=1$, $d=3$) and 0.63 ($n=2$, $d=3$) in good agreement with high-temperature series, and $a_m^-/a_\chi^+=0.85$ ($n=1$, $d=3$).

Several years ago it was realized both experimentally[1] and theoretically[2] that analyses of data taken near to a critical point should include "correction-to-scaling" confluent singular terms. A thermodynamic quantity f_i should therefore be written as[3]

$$f_i=A_i|t|^{-\lambda_i}[1+a_i|t|^\Delta+O(|t|^{2\Delta})], \tag{1}$$

where $t=(T-T_c)/T_c$. Experimental results, which are fitted to the pure power law $f_i\simeq A_{i,\text{eff}}\,t^{-\lambda_{i,\text{eff}}}$, will therefore yield values of $\lambda_{i,\text{eff}}$ as[4]

$$\lambda_{i,\text{eff}}=\lambda_i-a_i\Delta|t_i|^\Delta. \tag{2}$$

The leading critical exponents λ_i and correction-to-scaling exponent Δ are now known with high accuracy via $\epsilon=4-d$ expansion, supplemented by knowledge of the asymptotic behavior of perturbation series,[5] for example $\Delta=0.493$, 0.521, and 0.550 for $n=1$, 2, and 3 component systems, respectively[6] (d is the space dimensionality and n the dimensionality of the order parameter).

There is also considerable theoretical information available on the universal relations among the leading critical amplitudes[7-9]; however, only recently has much experimental[10,11] or theoretical attention been paid to the ratios among the correction amplitudes a_i. High-temperature series[12] suggest that a_ξ^+/a_χ^+ is universal (ξ and χ are the correlation length and susceptibility; the superscript "plus" means $T>T_c$) with values 0.70 ± 0.03 ($n=1$) and 0.6 ± 0.1 ($n=2$). The present authors[13] established the universality and calculated a_c^+/a_c^- (c is the specific heat; the superscript "minus" means $T<T_c$) to order ϵ^2; they found a value of 1.17 for $n=2$, $d=3$ which agrees quite well with Ahler's[4,10] experimental value of 1.29 ± 0.25 for ^{4}He on the lambda line. More recently Aharony and Ahlers[4] have observed that to leading order (zeroth order in ϵ) one has

$$a_i/a_j=(\lambda_i-\lambda_i^0)/(\lambda_j-\lambda_j^0), \tag{3}$$

where λ^0 are the mean-field values of the expo-