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Dynamic Confinement from Velocity-Dependent Interactions

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Consideration of a new class of interactions is proposed. These contain certain velocity-dependent terms which lead to confined sytems by not permitting scattering. The relativistic two-body system is discussed as an example. Applications to quark-quark interactions are conceivable.

Phenomenological models of mesons as quark-antiquark bound states are currently of great interest. The usual approach is to reduce the motion of a pair of quarks to an equivalent particle moving in an effective potential. This potential is chosen to be a function of the relative quark separation only and to increase in such a way that it confines the quarks dynamically.[1]

Quarks in hadrons are known to be relativistic. Therefore, velocity-dependent interactions cannot be ruled out. This is especially true when the phenomenological interaction is viewed as an effective interaction obtained from quantum chromodynamics (QCD) when all but a finite number of degrees of freedom are frozen. For this reason we have been studying N-body, relativistic, direct-interaction dynamics[2,3] and have been led to consider an unorthodox interaction that depends upon the particles' *relative* positions *and* momenta. We have solved the equations of motion of the two-body case and have found the class of interactions for which scattering solutions are not permitted.

There is another, more fundamental reason for studying direct-interaction dynamics. It has been proven that under rather general conditions[3] relativistic Hamiltonian direct-interaction dynamics for an N-particle system requires many-body forces in order to permit a cluster decomposition. Therefore if one insists on pairwise interactions only, a cluster decomposition seems impossible. A consistent direct-interaction theory can then be constructed only for confining forces. This is somewhat analogous to gauge quantum field theories where the infrared structure prevents a cluster decomposition and brings about confinement.[4]

We start with the general covariant N-body Hamiltonian dynamics of Ref. (2) and specialize to the case $\varphi_1=\varphi_2$ for two particles with unequal masses. The internal dynamics of the two-body system can be expressed in terms of the positions and momenta $\xi_a, \pi_a (a=1, 2)$ *relative* to the center of momentum (c.m.) with use of the Hamiltonian

$$H=\sum_{a=1}^{2} \chi_a/2m_a, \tag{1}$$

where

$$\chi_a=\vec{\pi}_a^{\,2}+m_{a0}^{\,2}+\varphi(\vec{\xi}_b, \vec{\pi}_b)\approx\eta_a+m_{a0}^{\,2},$$

$$m_a=(\eta_a+m_{a0}^{\,2})^{1/2},$$

and

$$\eta_a=\text{const}>0.$$

The χ_a are constants because $\pi_1+\pi_2=0$ in the c.m. frame and therefore $\dot{\chi}_a\approx 0$. The symbol $\approx$ means "weak equality" in the sense of Dirac.[5] The Hamiltonian generates evolution with respect to a parameter τ which is chosen to be the proper time of an observer comoving with the

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center of the system. $\vec{\xi}_a$ and $\vec{\pi}_b$ obey the canonical equal-time bracket relations:

$$\{\xi_a{}^k, \pi_b{}^l\} = \delta_{ab}\delta^{kl},$$
$$a, b = 1, 2; \quad k, l = 1, 2, 3. \tag{2}$$

Despite the fact that the kinetic energy term looks nonrelativistic, this is a relativistic Hamiltonian as can be seen by calculating the velocities in the noninteracting case $\varphi = 0$:

$$\dot{\vec{\xi}}_a = \{\vec{\xi}_a, H\} = \vec{\pi}_a/(\eta_a + m_{a0}{}^2)^{1/2},$$
$$\dot{\vec{\xi}}_a{}^2 = \frac{\vec{\pi}_a{}^2}{\eta_a + m_{a0}{}^2} \approx \frac{\eta_a}{\eta_a + m_{a0}{}^2} < 1.$$

The reduction of the motion from two particles to that of one equivalent particle is performed by working in the c.m. frame where the sum of the internal momenta vanishes; thus we define

$$\vec{\pi} \equiv \vec{\pi}_1 = -\vec{\pi}_2,$$
$$\vec{\xi} \equiv \vec{\xi}_1 - \vec{\xi}_2,$$

and therefore

$$\{\xi^k, \pi^l\} = \delta^{kl}.$$

The equivalent Hamiltonian is

$$H = \chi/2m, \tag{3}$$

where

$$\chi = \vec{\pi}^2 + m_0{}^2 + \varphi(\vec{\xi}, \vec{\pi}) \approx \eta + m_0{}^2,$$
$$m = \frac{(\eta + m_{10}{}^2)^{1/2}(\eta + m_{20}{}^2)^{1/2}}{(\eta + m_{10}{}^2)^{1/2} + (\eta + m_{20}{}^2)^{1/2}},$$
$$m_0{}^2 = (\eta + m_{10}{}^2)^{1/2}(\eta + m_{20}{}^2)^{1/2} - \eta,$$

and

$$\eta = \eta_1 = \eta_2;$$

m is a generalization of the reduced mass and m_0 is the rest mass of the equivalent particle. In general, both of these masses depend upon the interaction via η.

After these preliminaries we now turn to the proposed interaction. The prototype of the new class of interactions is

$$\varphi = -\beta^2(\vec{\xi}\cdot\vec{\pi})^2, \quad \beta = \text{const.} \tag{4}$$

The equations of motion of the equivalent particle are

$$\dot{\vec{\xi}} = m^{-1}[\vec{\pi} - \beta^2(\vec{\xi}\cdot\vec{\pi})\vec{\xi}]$$

and

$$\dot{\vec{\pi}} = m^{-1}[\beta^2(\vec{\xi}\cdot\vec{\pi})\vec{\pi}].$$

These imply the harmonic oscillator equations

$$\ddot{\vec{\xi}} = -m^{-2}\beta^2[\pi^2 - \beta^2(\vec{\xi} - \vec{\pi})^2]\vec{\xi} \approx -\omega^2\vec{\xi},$$

where

$$\omega^2 \equiv m^{-2}\beta^2\eta.$$

The equations of motion have as solution

$$\vec{\xi} = \hat{i}\beta^{-1}\sin\omega\tau - \hat{j}J\eta^{-1/2}\cos\omega\tau,$$
$$\vec{\pi} = \hat{i}\eta^{1/2}/\cos\omega\tau.$$

The unusual energy dependence in ω and the solution for $\vec{\pi}$ both emphasize differences between this solution and that of the well-known isotropic three-dimensional harmonic oscillator. The constant angular momentum vector has been used to choose the x-y plane as the arena of motion ($\vec{J} = J\hat{k}$).

The meaning of the strange τ dependence of $\vec{\pi}$ is the following. Since the "no-interaction theorem" of Currie, Jordan, and Sudarshan[6] it has been known that the canonical variables are not always to be identified with the physical variables, and may in fact not be simply related to them. Here the canonical $\vec{\pi}$ diverges twice each period while the (relativistic) physical momentum, $\vec{\pi}_{\rm phys} = m\dot{\vec{\xi}}$, is well behaved. In terms of this physical momentum, the Hamiltonian (3) with the interaction (4) has the form

$$H = \frac{\vec{\pi}_p{}^2 + m_0{}^2}{2m} + \frac{\beta^2}{2m}\frac{(\vec{\xi}\cdot\vec{\pi}_p)^2}{(1 - \beta^2\vec{\xi}^2)} \approx \frac{\eta + m_0{}^2}{2m}.$$

Of course, $\vec{\xi}$ and $\vec{\pi}_{\rm phys}$ are not canonical conjugates:

$$\{\xi^k, \pi_p{}^l\} = \delta^{kl} - \beta^2\xi^k\xi^l,$$
$$\{\pi_p{}^k, \pi_p{}^l\} = -\beta^2(\vec{\xi}\wedge\vec{\pi})^{kl}.$$

In view of the divergence of the canonical $\vec{\pi}$, we have solved the above system in Hamilton-Jacobi form as a check on the internal consistency of the theory. The principal function S is found to be differentiable throughout the entire motion. Therefore, the canonical momentum is single valued and well defined.

Having thus established that the unconventional interaction (4), which depends on the canonical momentum, results in a confined motion similar to that of an isotropic harmonic oscillator, we considered the following generalization:

$$\varphi(\vec{\xi}, \vec{\pi}) = (\vec{\xi}\cdot\vec{\pi})^2 B(\vec{\xi}^2) + V(\vec{\xi}^2). \tag{5}$$

The equations of motion that follow from (5) are

$$\dot{\vec{\xi}} = m^{-1}[\vec{\pi} + (\vec{\xi}\cdot\vec{\pi})B\vec{\xi}], \tag{6a}$$

$$\ddot{\vec{\xi}} = -(1/m^2\vec{\xi}^4)(F + J^2 - \vec{\xi}^2 F')\vec{\xi}, \tag{6b}$$

$$F(\vec{\xi}^2) \equiv (1 + B\vec{\xi}^2)[(\eta - V)\vec{\xi}^2 - J^2], \tag{6c}$$

where the prime denotes the derivative with respect to $\vec{\xi}^2$. For finite $\vec{\xi}^2$, the solutions are periodic and lie in distinct physical regions defined by

$$1 + B\vec{\xi}^2 \geq 0, \tag{7a}$$

$$\eta - V - J^2/\vec{\xi}^2 \geq 0. \tag{7b}$$

The number and size of these intervals depend upon the detailed form of B and V. B and V are restricted by a relativity condition (necessary for velocity-dependent interactions),

$$(1/\vec{\xi}^2)(F + J^2) < 4m^2, \tag{8}$$

to ensure that $\dot{\vec{\xi}}$ be the spatial components of a *timelike* four-vector.

The solutions of the generalized interaction (5) have been examined for scattering solutions. The general solution partitions into five cases depending upon the large-$\vec{\xi}^2$ behavior $F_\infty(\vec{\xi}^2)$ of $F(\vec{\xi}^2)$:

(A) $F_\infty(\vec{\xi}^2) < 0$. There is no physical region for large $\vec{\xi}^2$ and therefore no scattering. The system is confined. This case includes the prototype interaction.

(B) $F_\infty(\vec{\xi}^2) > 0$, $F_\infty(\vec{\xi}^2) \to 0$. These are "pseudoscattering" solutions in which the equivalent particle approaches infinity, slows down, and stops.

(C) $F_\infty(\vec{\xi}^2) > 0$, $F_\infty(\vec{\xi}^2) \sim (\vec{\xi}^2)^\alpha$, $0 \leq \alpha < 1$. The Hamilton-Jacobi principal function S has a cusp. It is continuous but not differentiable and, as a result, the canonical momentum is not single valued throughout the motion.

(D) $F_\infty(\vec{\xi}^2) > 0$, $F_\infty(\vec{\xi}^2) \sim \vec{\xi}^2$. These are (genuine) scattering solutions where the motion is that of a free particle for large $|\vec{\xi}|$. This motion can arise only if both B and V fall off fast enough:

$$B_\infty(\vec{\xi}^2) \sim (1/\vec{\xi}^2)^\lambda, \quad \lambda \geq 1,$$

$$V_\infty(\vec{\xi}^2) \sim (1/\vec{\xi}^2)^\rho, \quad \rho \geq 0,$$

for large $\vec{\xi}^2$. Scattering from a short-range potential with no velocity-dependent term ($B = 0$, $V \to 0$) is a trivial example of this case.

(E) $F_\infty(\vec{\xi}^2) > 0$, $F_\infty(\vec{\xi}^2) \sim (\vec{\xi}^2)^\alpha$, $\alpha > 1$. $(1/\vec{\xi}^2)^{-1} \times F_\infty(\vec{\xi}^2)$ diverges for large $\vec{\xi}^2$ and therefore violates the relativity condition (8).

Of these five cases only two are acceptable as physical solutions—cases (A) and (D). Case (E) has already been excluded on physical grounds. Case (C) is unacceptable since the canonical momentum must be single valued in order that the theory be self-consistent. In addition, solutions of case (C) and of case (B) can be excluded from general consideration because they can arise only in the very special case when the leading term of $B_\infty(\vec{\xi}^2)$ is precisely $-(1/\vec{\xi}^2)$ for large $\vec{\xi}^2$.

The solutions of cases (A) and (D) lie in distinct physical regions $F(\vec{\xi}^2) \geq 0$. These solutions are distinct because if the equivalent particle initially lies in one region it will remain in that region. If it is a finite region [case (A) or (D)], the particle will undergo periodic motion. If it is an unbounded region [case (D) only] then the particle will undergo scattering. Cases (A) and (D) can exhibit an angular momentum barrier [$F(0) = -J^2 < 0$] only if

$$B(\vec{\xi}^2) \sim (1/\vec{\xi}^2)^\lambda, \quad \lambda < 1, \quad V(\vec{\xi}^2) \sim (1/\vec{\xi}^2)^\rho, \quad \rho < 1,$$

for small $\vec{\xi}^2$. This is consistent with case (D) (scattering) only if $B = 0$ and $0 \leq \rho < 1$.

Concentrating on the confining case, (A), consider the interaction (5) when $B = -\beta^2 < 0$ and $V(\vec{\xi})$ vanishes for large $|\vec{\xi}|$. Asymptotically, then,

$$F_\infty(\vec{\xi}^2) = (-\eta\beta^2)\vec{\xi}^4.$$

The solution is confining [case (A)] if $\eta > 0$ and is unphysical [case (E)] otherwise.

Limits of the motion are given by the positivity conditions (7a) and (7b). Equation (7a) gives the upper bound of the motion $\vec{\xi}^2 < 1/\beta^2$ while the lower bound is given by (7b),

$$\eta = (V + J^2/\vec{\xi}^2)|_{(\vec{\xi}^2)_{\min}}.$$

The form of V is restricted by the positivity of η in (7b) and by the relativity condition (8). These restrictions require that $V(\vec{\xi}^2)$ be positive for all physical $\vec{\xi}^2$.

These classical considerations can now be extended to relativistic quantum mechanics. The canonical quantization of (3) is straightforward and the ambiguity in ordering of the first term in (5) (we assume again $B = -\beta^2$) only involves a trivial constant (see below). In the c.m. frame the relativistic Schrödinger equation looks nonrelativistic:

$$[\nabla^2 - \beta^2(\vec{\xi}\cdot\nabla)_{\text{ord}}{}^2 - V(\vec{\xi}^2)]\psi = -\eta\psi. \tag{9a}$$

For small ξ one can assume that the confining interaction is negligible compared with V (e.g., if V is of the Coulomb type). For large $|\vec{\xi}|$, if V vanishes and therefore gives scattering states in the absence of the confining interaction, one now obtains a square-integrable wave function because of $\beta^2 \neq 0$.

An expansion in spherical harmonics of the type $R_l Y_l/\xi$ leads to a radial function

$$R_l(\xi) = (\beta\xi)^{l+1} u_l((\beta\xi)^{1/2}),$$

where $u_l(t)$ satisfies

$$t(1-t)u_l'' + [l + \tfrac{3}{2} - t(l - \tfrac{5}{2})]u_l' + \tfrac{1}{4}[\nu(\nu+1) - w - (l+1)(l+2)]u_l = 0 \quad (9b)$$

with $w(t) = V(\xi^2)/\beta^2$, and $\nu(\nu+1) = c + \eta/\beta^2$. The constant c depends on the choice of ordering and is conveniently chosen to be $-\frac{1}{4}$. Asymptotically, for large ξ, u_l becomes a hypergeometric function,

$$u_l(t) \sim t^{-1-(l+\nu)/2} {}_2F_1(\tfrac{1}{2}(l+\nu+2), \tfrac{1}{2}(\nu - l + 1), \nu + \tfrac{3}{2}, t^{-1}) \quad (10)$$

which is in $L^2([0, \infty), t^{l+1/2}dt)$ for $\nu > -\frac{1}{2}$. The details of the spectrum depend, of course, on the choice of V.

The result (10) shows that the quantum-mechanical confinement produced by (4) is weak; the wave functions do not fall off exponentially, but only like a power. Application to quark binding is therefore expected to include additional confining interactions contained in V. But there is no reason to exclude interactions of the form (4) from a relativistic phenomenological description.

In summary, we can say the following:

(a) In a relativistic particle theory velocity-dependent interactions cannot be excluded. For the two-body system the form (5) of the interaction is essentially unique: Higher powers in $\vec{\pi}$ would violate the second-order character of the Schrödinger equation, and terms linear in $\vec{\pi}$ can be transformed away by a gauge transformation.

(b) The velocity-dependent term can provide confinement under suitable conditions [case (A) above]. In the quantized theory this confinement is weak. Quark binding is thus expected to involve another confining interaction, too, such as the conventional linear or logarithmic type.

(c) The proposed interaction term depends on a canonical momentum which is not the physical momentum. A similar difference between canonical and physical *position* has been earlier in the context of the "no-interaction theorem".[6]

(d) The proposed velocity-dependent interaction arises from a certain "gauge choice" of the Hamiltonian H in relativistic N-particle dynamics, viz. when no constraints are admitted in H. This is discussed in detail elsewhere.[7]

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