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Perturbative Contributions to Quark Masses

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The contribution of lowest-order electromagnetic (or weak) corrections to quark masses in quantum chromodynamics is considered. It is found that each contribution to the running mass is calculable from physics well below the grand unified scale, as long as the number of quark flavors n_f is greater than or equal to eleven. The renormalization-group expression for the running mass is also derived, by use of Dyson's equation for the self-energy of the quark.

In this Letter we shall consider the question of the convergence of the lowest-order electromagnetic (or weak) contributions to quark and hadronic masses[1] from the perspective of asymptotically free quantum chromodynamics. With use of the standard operator-product analysis, it is clear that the only contributions to the electromagnetic shift of the masses of hadrons which are potentially ultraviolet divergent are those associated with the perturbative contributions to the quark masses.[2]

Surprisingly, we find that if there exist at least eleven quark flavors, i.e.,

$$\tfrac{21}{2} < n_f < \tfrac{33}{2}, \tag{1}$$

then the lowest-order electromagnetic and weak contributions to the quark masses are individually finite and in principle calculable from physics well below the grand unification scale. The origin of this result is the fact that if $n_f > \frac{21}{2}$, then the running mass of quantum chromodynamics (QCD) decreases asymptotically faster than a logarithm, thus ensuring the convergence of a self-energy integral.

From the standpoint of a unified theory of strong, weak, and electromagnetic interactions, the consideration of individual perturbative contributions to the quark masses may in some case seem irrelevant. For example, Weinberg[3] has remarked that in models where after spontaneous symmetry breaking the zeroth-order contribution to a certain mass vanishes, but where the unbroken symmetries still allow the appearance of this mass in higher orders, the renormalizability of the theory guarantees that the sum of all perturbative contributions is finite in every order of the unified coupling constant.

In the previous case, however, the cancellation and consequent ultraviolet convergence are not expected to occur until momenta of the order of the grand unification scale ($p \sim m_0 \sim 10^{15}$ GeV). In contrast, if there in fact do exist eleven quark flavors, then the extra asymptotic convergence of QCD renders the lowest-order electromagnetic and weak contributions to quark masses individually convergent at a momentum scale of the order of the eleventh quark-flavor threshold, presumably well below the region where grand-unification effects or new dynamical couplings of the quarks must be taken into account. We should remark, however, that models with dynamical symmetry breaking contain naturally a soft chirality-mixing insertion, leading to an effective ultraviolet cutoff which can be made much smaller than the grand unification scale.[4]

The proper self-energy of a quark can be readily computed using the Dyson equation illustrated in Fig. 1, which reads

$$\Sigma(p)=\int\frac{id^4k}{(2\pi)^4}g_0^2\gamma_\mu D^{\mu\nu}(k)S(p-k)\Gamma_\nu(p-k,p) \tag{2}$$

in terms of unrenormalized vertices and propagators. Note that the renormalization of this equation has to deal with the problem of overlapping divergences. Using the Dyson equation for the vertex, one can express $Z_1\gamma_\mu$ in terms of the renormalized vertex function, and then write $Z_2^F\Sigma(p)$ in terms of renormalized quantities inside the integral, with no overlapping divergence.[5]

We define the "running" mass $m(p^2)$ as the renormalized mass parameter in the off-shell quark propagator,

$$S'(p)=\frac{1}{\not p-m_0-\Sigma(p)+i\epsilon}=\frac{Z(p^2)}{\not p-m(p^2)+i\epsilon}\equiv Z_2^F\tilde{S}'(p), \tag{3}$$

and then use Eq. (2) in the Landau gauge, where the separation of wave-function and mass renormalizations is simple because the first one is trivial ($Z_2^F=1$). For QCD we get, for large p^2 and in leading-logarithm approximation, the homogeneous equation $[C_F=(n_c^2-1)/2n_c=\frac{4}{3}]$

$$m_s(p^2)=\frac{3}{4\pi}C_F\int_{p^2}^{\infty}\frac{dk_2}{k^2}\alpha_s(k^2)m_s(k^2), \tag{4}$$

where $\alpha_s(k^2)\cong 4\pi/(\beta\ln k^2/\Lambda^2)$, with $\beta=11-\frac{2}{3}n_f$, is the running coupling constant at large k^2.[6] We can also express this result as an evolution equation,

$$\frac{\partial}{\partial\ln p^2}m_s(p^2)=-\frac{3}{4\pi}C_F\alpha_s(p^2)m_s(p^2). \tag{5}$$

The solution to Eqs. (4) and (5) is

$$m_s(p^2)=m_s(p_0^2)[\alpha_s(p^2)/\alpha_s(p_0^2)]^{3C_F/\beta}. \tag{6}$$

This result, which is valid for $|p^2|\gg m_f^2$ (the heaviest quark threshold), is conventionally derived using renormalization-group methods,[7] and is valid for general covariant gauges.[8] Here p_0^2 is a normalization point which is often chosen at the grand unification scale.

Let us assume that the running mass $m_s(p^2)$ for strong interactions has been specified, including its normalization. We can then consider the lowest order $[O(\alpha)]$ perturbation $\delta m(p^2)$ to the running mass due to electromagnetic interactions. Provided the integrals are convergent, we have, for large p^2,

$$\delta m(p^2)=\frac{3}{4\pi}\int_{p^2}^{\infty}\frac{dk^2}{k^2}[e_q^2\alpha m_s(k^2)+C_F\delta\alpha_s(k^2)m_s(k^2)+C_F\alpha_s(k^2)\delta m(k^2)]. \tag{7}$$

The three terms can be identified with the Dyson-equation contributions indicated in Figs. 2(a)–2(c), respectively.

The change in the QCD running coupling constant $\delta\alpha_s(k^2)$ due to lowest-order electromagnetic interactions corresponding to Fig. 2(b) is of order $\alpha\alpha_s(k^2)$. The $\delta\alpha_s(k^2)m_s(k)$ term thus can be neglected at large k^2 compared with the $e_q^2\alpha m_s(k^2)$ term in Eq. (7).

The central question is the ultraviolet convergence of the integral equation (7) for $\delta m(p^2)$. We note that if $3C_F/\beta>1$, i.e., $n_f>\frac{21}{2}$, then $(\ln k^2)m_s(k^2)\to 0$ for $k^2\to\infty$ [see Eq. (6)] and the integral of the

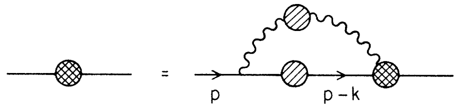

FIG. 1. Dyson equation for the self-energy. Double-shaded blocks indicate irreducible self-energy and vertex insertions.

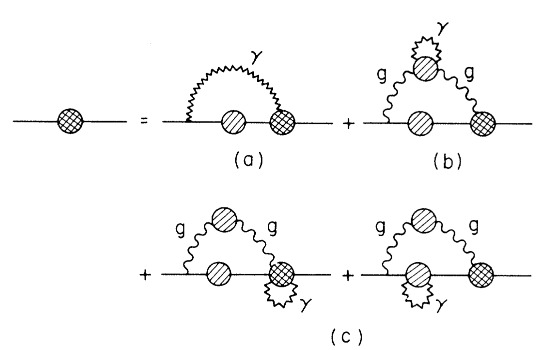

FIG. 2. Dyson equation for computing order-α corrections to quark masses in QCD.

first term of Eq. (7) is convergent. In fact if $3C_F/\beta>1$, the solution to Eq. (7) which is proportional to α is[9]

$$\delta m(p^2) = -\frac{3}{4\pi}\alpha e_q^2 m_s(p^2)\ln\left(\frac{p^2}{\Lambda^2}\right), \tag{8}$$

i.e.,

$$\frac{\delta m(p^2)}{m_s(p^2)} = -\frac{3}{\beta}\frac{\alpha e_q^2}{\alpha_s(p^2)}. \tag{9}$$

Equation (8) is valid for $|p^2| \gg m_f^2$, where m_f^2 is the threshold for the eleventh flavor threshold. Thus if there are at least eleven (but not more than sixteen) quark flavors,[10] the lowest-order electromagnetic and weak interaction contributions to the running quark mass are each finite and in principle calculable in QCD. In particular, the order-α contribution to the "bare" mass of the total Lagrangian $\lim_{p^2\to\infty}\delta m(p^2)$ vanishes.

Note that the complete electromagnetic contribution to the quark mass to order α at the hadronic mass scale requires a detailed calculation of the small-$|k^2|$ integration region. The result of Eq. (9) shows that the *large*-$|k^2|$ region of integration, where $\beta<4$, gives a *negative* contribution to the quark mass; i.e., this contribution tends to make the u quark *lighter* than the d quark: $\delta m_u(p^2) < \delta m_d(p^2)$. However, we emphasize that the calculation of $m_u - m_d$ (or $m_p - m_n$) still requires knowledge of the low-momentum region as well as the weak-interaction contributions. Furthermore it is not clear that the u- and d-quark masses are degenerate in the absence of electromagnetic or weak-interaction contributions $[m_s^u(p_0^2) \overset{?}{=} m_s^d(p_0^2)]$.

It is interesting to compare the result of Eq. (9) with the corresponding renormalization-group result for the running mass.[7] If we consider only QCD and electromagnetic interactions, then for large p^2 we have

$$\frac{\partial m(p^2)}{\partial \ln p^2} = -\frac{3}{4\pi}\left[C_F\alpha_s(p^2) + e_q^2\alpha(p^2)\right]m(p^2), \tag{10}$$

i.e., the normalized running mass is (with the one-loop approximation to the renormalization-group β functions for α and α_s),

$$m(p^2) = m_0\left[\frac{\alpha_s(p^2)}{\alpha_s(p_0^2)}\right]^{3C_F/\beta}\left[\frac{\alpha(p^2)}{\alpha(p_0^2)}\right]^{-(9e_q^2/4K_f)}, \tag{11}$$

where $m_0 = m(p_0^2)$ and $\alpha(p^2)$ is the quantum electrodynamics (QED) running coupling constant,[11]

$$\frac{\alpha(p^2)}{\alpha(p_0^2)} \cong \left(1 - K_f\frac{\alpha(p_0^2)}{3\pi}\ln\frac{p^2}{p_0^2}\right)^{-1}, \tag{12}$$

and $K_f = \sum_f e_f^2$, the sum of the squares of the charges of all the fermions ($K_f = \frac{4}{3}n_f$ in the usual generation replication schemes).

Thus

$$\delta m(p^2) = m(p^2) - m_s(p^2) = m_s(p^2)\left[\left(\frac{\alpha(p^2)}{\alpha(p_0^2)}\right)^{-(9e_q^2/4K_f)} - 1\right] + \frac{m_0 - m_{0s}}{m_0}m(p^2) \tag{13}$$

where, in general, we expect $m_s(p_0^2) \neq m(p_0^2)$ (even at the grand unification scale). If $0<\beta<4$, then we can use Eq. (9) and

$$(m_0 - m_{0s})/m_{0s} = -(3/\beta)\alpha e_q^2/\alpha_s(p_0^2) \tag{14}$$

to specify Eq. (13) to lowest order in $\alpha = \alpha(p_0^2)$. Although these results cannot be trusted quantitatively when $\alpha_s(p^2)$ is of order α, we see that $\delta m(p^2)/m(p^2)$ becomes of order 1 as one approaches the grand unification scale.

In conclusion, we have found that the strong asymptotic-freedom convergence of QCD with $16 \geq n_f \geq 11$ quark flavors is sufficient to render the lowest-order and electromagnetic contributions to quark masses calculable from integrals involving the QCD and quark mass scales alone. In general, perturbative terms of order α, α^2, ..., α^n are all calculable if $\beta < 4/n$; i.e., $n_f > \frac{3}{2}(11 - 4/n)$ (e.g., the terms of order α, α^2, ..., α^{11} are calculable if $n_f = 16$). The higher-order terms involve contributions of order $\alpha^{n+1}(\ln\Lambda_0^2/m^2)^{n+1-4/\beta}$ where Λ_0^2 is an ultraviolet cutoff. Assuming this convergence is set by a grand unification scale where $\ln\Lambda_0^2/m^2 \ll \alpha^{-1}$, then the higher-order terms are still relatively small.

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[1]In this paper we analyze the electromagnetic and weak-interaction contributions to the (renormalized) quark running mass $m(p^2)$ as well as to the bare mass parameter $\lim_{p^2\to\infty} m(p^2)$ in the total Lagrangian. In contrast, the Cottingham formula yields the order-α perturbation to the quark or hadron mass renormalized only by the strong interactions. [See W. N. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963); W. I. Weisberger, Phys. Rev. D 5, 2600 (1972); A. Zee, Phys. Rep. 3C, 129 (1972).] This quantity is logarithmically divergent and requires renormalization. See J. C. Collins, Nucl. Phys. B149, 90 (1979), and B153, 546(E) (1979). G. B. West, Los Alamos Scientific Laboratory Report No. LA-UR-79-1690 (to be published); J. Kiskis, private communication. We wish to thank M. Dine, G. P. Lepage, and K. Johnson for helpful discussions on this point.

[2]The regularity noted by Harari that the $\Delta I = 2$ mass differences such as $m_{\pi^\pm} - m_{\pi^0}$ or $m_{\Sigma^+} + m_{\Sigma^-} - 2m_{\Sigma^0}$ can be computed in terms of a dispersion sum over low-lying resonances, but that $\Delta I = 1$ mass differences such as $m_p - m_n$, $m_{K^+} - m_{K^0}$, $m_{\Sigma^+} - m_{\Sigma^0}$, $m_{\Xi^+} - m_{\Xi^0}$ cannot, is due to the fact that the quark-mass contributions cancel for $\Delta I = 2$ mass differences. See H. Harari, Phys. Rev. Lett. 17, 1303 (1966).

[3]S. Weinberg, Phys. Rev. Lett. 29, 388 (1972).

[4]S. Dimopoulos and L. Susskind, Institute for Theoretical Physics, Stanford University, Report No. ITP-626-Stanford, 1979 (to be published).

[5]This method is used in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw Hill, New York, 1965). For a more general approach see M. Baker and C. Lee, Phys. Rev. D 15, 2201 (1977).

[6]We neglect contributions to $m_s(p^2)$ beyond one-loop since these are higher order in $\alpha_s(p^2)$. Note that by using the Ward identity for the $\gamma q\bar{q}$ vertex, we can calculate $\partial m_s/\partial p^2$ directly as a function of renormalized quantities, with an overlapping-divergence-free skeleton expansion. Thus

$$\frac{\partial m_s(p^2)}{\partial \ln p^2} = \alpha_s(p^2) m_s(p^2) f\left(\alpha_s(p^2),\ \frac{m_s^2(p^2)}{p^2}\right),$$

and for large p^2, we only require $f(0,0)$.

[7]H. Georgi and H. D. Politzer, Phys. Rev. D 14, 1829 (1976); A. J. Buras, J. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B135, 66 (1978).

[8]See for example, O. Nachtmann and W. Wetzel, Nucl. Phys. B146, 273 (1978).

[9]Formally, the solution to Eq. (7) is given by the order-α contribution given in Eq. (8) plus a term $Am_s(q^2)$, which is the general solution to the homogeneous part of the integral equation

$$\delta m(p^2) = \frac{3}{4\pi}\int_{p^2}^{\infty} \frac{dk^2}{k^2} C_F \alpha_s(k^2)\,\delta m(k^2)$$

obtained by setting $\alpha \equiv 0$ in Eq. (7). Since this term has no dependence on α, such a contribution should be incorporated into the definition and normalization of $m_s(p^2)$ rather than the order-α electromagnetic perturbation (i.e., $A = 0$).

[10]Alternatively, one could have other irreducible representations of color SU(3) which yield $0<\beta<4$.

[11]If we assume that QED is imbedded in a grand unified theory which is asymptotically free, the effects of unification will prevent any singularity in $\alpha(p^2)$. See H. Georgi and S. Glashow, Phys. Rev. Lett. 32, 438 (1974), and Ref. 7.

Analytic Calculation of Higher-Order Quantum-Chromodynamic Corrections in e^+e^- Annihilation

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Order-α_s^2 corrections to the annihilation of e^+e^- into hadrons are computed analytically. We briefly discuss the technique, which involves the extension to noninteger dimensions of Chebyshev expansions. In the energy range 15–30 GeV (five flavors) we find

$$R = 3\sum e_Q^2\{1 + \alpha_s(q)/\pi - 0.94[\alpha_s(q)/\pi]^2 + \ldots\},$$

where α_s is the momentum-space subtracted strong-coupling constant. Our result agrees with that of Dine and Sapirstein, and Chetyrkin, Kataev, and Tkachov.

The calculation of high-order quantum-chromodynamic (QCD) corrections to $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is of particular significance in at least two respects. First of all, both the theoretical and experimental determinations of R involve fewer parameters than an analysis of deep-inelastic

