VOLUME 44, NUMBER 7 PHYSICAL REVIEW LETTERS 18 FEBRUARY 1980

Possible Role of Incipient Anderson Localization in the Resistivities of Highly Disordered Metals

Yoseph Imry
Physics Department, Brookhaven National Laboratory, Upton, New York 11973, and Department of Physics and Astronomy,[(a)] *Tel Aviv University, Ramat Aviv, Israel*

(Received 25 September 1979)

The scaling theory of Anderson localization by Abrahams *et al.* is used for dirty conductors just above the Anderson transition. When the inelastic mean free path, l_{ph}, is smaller than the coherence length, ξ, in the extended phase, the conductivity *increases* with temperature like l_{ph}^{-1}. This may be related to the quite general correlation between large resistances and their negative temperature derivatives found by Mooij. $l_{\text{ph}} > \xi$ is required to distinguish between the extended and localized regimes.

Well-known anomalies,[1,2] unexplainable by conventional transport theory, exist in highly disordered metals and alloys. For example, once the zero-temperature resistivity of *A*15 metals exceeds an order of magnitude of ~ 150 $\mu\Omega$ cm, the resistivity *decreases* with increasing temperature to a high-temperature "saturation" value which is only weakly dependent on disorder.[1] The latter "saturation" effect, which occurs also for clean *A*15 metals, probably depends also on specific properties of these transition metals, such as having narrow bands.[3] The breakdown of Mathiessen's rule, $d\rho/dT$ being negative for large enough ρ, appears to be quite universal as emphasized by Mooij[2] who found such correlations between $d\ln\rho/dT$ and ρ for a large class of materials. It is also possible that the decrease in the resistivity of liquid metals with increasing temperature may be sometimes related to the same presumably general phenomenon.

It is not hard to understand why conventional transport theory may fail once the electronic mean free path, l, becomes only somewhat larger than the microscopic distance a (l is probably ~ 10–20 Å for $\rho \sim 10^{-4}$ Ω cm). In fact $l \sim a$ is the well-known Ioffe-Regel criterion,[4] where, as emphasized by Mott,[5] one would expect localization[6,7] or at least a breakdown of perturbation expansions in which a/l is a small parameter. In the case $l \gtrsim a$, localization may have not yet occurred but nevertheless it is reasonable to start from the localization theory rather than from the usual $l \gg a$ approximations.[3] A similar point has already been made by Jonson and Girvin[8] who performed numerical calculations for an Anderson model on a Cayley tree, which produced a qualitative agreement with the Mooij[2] correlation. These calculations, however, also yielded a "minimum metallic conductivity," which does not occur in the scaling theory[9,10] in which the resistance diverges, at $T = 0$, near the Anderson transition.

While attempts[11] to formulate the microscopic theory of Anderson localization in analogy with known phase transitions have suffered from significant drawbacks,[10] a renormalization-group scaling theory[9] of localization was recently constructed by Abrahams *et al.*[10] relying on very general arguments. The predictions of this theory, taking also into account inelastic scattering of the electrons by phonons[12] and heating effects,[13] appear to agree with experiments on effectively two-dimensional (2D) films.[14] The limiting case of 2D is, however, the most difficult one theoretically.[15] In the 3D case discussed here the qualitative validity of the theory[10] is quite clear.

The point that I wish to make in this note is the following. The scaling theory of localization involves a characteristic length, ξ, which diverges as the localization edge is approached from both the localized and the extended sides. The physical meaning of ξ in the extended phase will be discussed below. We assume that the material is so disordered that it is close enough to the localization edge so that ξ is much larger than both the elastic mean free path l_{el} and a. When the temperature is high enough so that ξ is much larger than the inelastic mean free path, l_{ph}, the latter[12] length scale will determine the conductivity, σ. It follows then as we shall see that $\sigma \propto l_{\text{ph}}^{-1}$ (taking the bulk, 3D case), i.e., σ *increases* with temperature. It is perhaps not surprising that σ increases with temperature for localized states where "hopping" conductivity is essential. However, we argue that this can also be true in the *extended phase*, provided only that $l_{\text{ph}} < \xi$. In fact, very generally, for a phase transition with a diverging correlation length, ξ, a measurement done on a length scale $\lesssim L$ sees the phase transition smeared over a range $T_c \pm \Delta T$, where $\xi(\pm \Delta T_c) \sim L$. Thus, for $l_{\text{ph}} < \xi$, σ cannot distinguish between localized or extended electronic states.

Let us review the renormalization-group-scaling[9, 10] picture for localization at $d = 3$ (the generalization to any $d > 2$ is straightforward). Abrahams *et al.*[10] focus attention on the way the conductance g (measured in units of $e^2/2\hbar \sim 1.2\times 10^{-4}$ mho) depends on the length scale L at $T = 0$. Compelling physical arguments[7, 10] suggest that for $T = 0$ and no interactions, g is the only relevant parameter. $g(L)$ can be thought of as the conductance of a cube of size L, once $L \gg l_{\mathrm{el}}$. g can also be defined[7] as the ratio between the appropriate two coupling constants: $\hbar$ divided by the diffusion time and the level spacing $dE/dN(E)$, both at scale L. The above ratio is also proportional to the Anderson's model[6] "V/W" ratio. In the conducting, "Ohmic" phase one expects $g(L) \propto L$, while in the localized phase g is expected to vanish exponentially with L. This leads to the following limits for the function $\beta(g) \equiv d\ln g/d\ln L$:

$$\beta(g) \to \begin{cases} 1, & \text{large } g, \\ -\text{const} + \ln g, & \text{small } g. \end{cases} \tag{1}$$

Since $\beta(g)$ is continuous (and in fact analytic) and should be *monotonic*, β must vanish at a certain point g_c. g increases (decreases) with L for $g > g_c$ $(g < g_c)$. $g = g_c$ is an (unstable) fixed point, corresponding to an L-independent g and a vanishing *conductivity*, σ, for large L. Thus, $g > g_c$ is the conducting phase and $g < g_c$ the insulating (localized) one. For $g \cong g_c$, $\beta(g)$ has a finite slope:

$$\beta(g) \cong s\ln(g/g_c) \equiv s\epsilon, \quad \epsilon_0 \equiv \ln(g/g_c) \ll 1, \tag{2}$$

where s is a number larger than[10] but of the order of unity. If it is assumed now that at some microscopic length scale, L_0, $g(L_0) = g_0 \cong g_c$, one can integrate (2) and find

$$g(L) \equiv g_c \exp[\epsilon_0 (L/L_0)^s], \quad \epsilon_0 \equiv (g_0 - g_c)/g_c, \tag{3}$$

which is valid only as long as (2) applies, but β has to go continuously to (1) at large $|\ln(g/g_c)|$. Thus (2) breaks down once at $s\epsilon \cong 1$ and goes over to $\beta = 1$ for the conducting phase and to the logarithmic dependence on g in the localized phase. The latter yields the exponentially vanishing g for $\epsilon < 0$. Focusing on the conducting phase, one sees that the crossover to the normal Ohmic behavior is obtained once

$$g \cong g_c e^{1/s} \sim g_c \times O(1), \quad L \cong L_0/\epsilon_0^{1/s} \equiv \xi(\epsilon_0). \tag{4}$$

For length scales larger than ξ, *the behavior is Ohmic* ($g \propto L$). It is important that g does *not* change in order of magnitude in the whole range $L_0 \lesssim L \lesssim \xi$. This follows from the simple assumptions (as in Ref. 10) that $s = O(1)$ and ρ is a smooth function. Evaluating the macroscopic conductivity $\sigma \cong g(\xi)/\xi$, it was found in Ref. 10 that, in agreement with Ref. 9,

$$\sigma = A g_c (e^2/\hbar\xi), \tag{5}$$

where A is a constant of order unity, $\sigma \to 0$ at the transition, and there is no "minimum metallic conductivity." It is important to understand the physical meaning of the length ξ. It signifies the scale at which the conductance becomes Ohmic (the resistance of a cube of side L goes like L^{-1}). It is remarkable that ξ diverges at the $(T = 0)$ transition to the insulating state, so that only samples of increasing sizes can be regarded as macroscopic as the material "tends to an insulator." In the localized side, ξ measures the localization range, where the wave functions decay to zero for lengths larger than ξ. In the extended phase the wave function, ψ, may be envisaged to have the following structure: $|\psi|$ is large around the point where it is going to be localized and decays within a length scale ξ to a small (possibly oscillating) value, which vanishes as $\xi \to \infty$ and the localization edge is approached. Thus, $\sigma(L)$ is clearly L dependent for $L < \xi$.

The above scaling picture is valid only when $\xi \gg \max(l_{\mathrm{el}}, a) \equiv L_0$ where a is a microscopic length. When the disorder is small enough so that $\xi \lesssim L_0$ the length L_0 becomes the relevant length. We note that a and hence L_0 can be larger than l_{el} for, e.g., a granular metal. We expect that $\xi \lesssim L_0$ is the condition for the validity of conventional transport theory at $T = 0$.

Now, suppose that the temperature T is finite, so that the inelastic mean free length l_{ph} is defined.[12] The $T = 0$ theory can now be used only at length scales $L \lesssim l_{\mathrm{ph}}$ (for $L \ll l_{\mathrm{ph}}$, inelastic collisions are not important). The temperature should thus have a small effect as long as it is so low that $l_{\mathrm{ph}} \gg \xi$. However, once T is such that

$$l_{\mathrm{ph}} < \xi, \tag{6}$$

then (5) is invalid, and the conductivity becomes sensitive to the inelastic effects. From (4), the *conductivity* at length scale l_{ph} $(L_0 < l_{\mathrm{ph}} < \xi)$ is

$$\sigma = B g_c (e^2/\hbar l_{\mathrm{ph}}), \tag{7}$$

where B is a constant of order unity. It now appears to make sense to further assume that l_{ph} *is the relevant length scale for the conductivity*,[12, 13] i.e., that the behavior is Ohmic $(g \propto L)$ for $L > l_{\mathrm{ph}}$. This should be true because the wave functions are scrambled enough by scattering over the length scale l_{ph}.

What this picture predicts, therefore, is that once l_{el} is so small that $\xi \gg L_0$ (i.e., roughly for $\rho \gtrsim 10^{-4}\ \Omega\,\mathrm{cm}$), ρ should decrease with increasing temperature for T larger than a characteristic temperature determined by (6). This characteristic temperature, T_0, will decrease with increasing disorder [increasing $\rho(T=0)$] and with increasing electron-phonon coupling. In fact, $T_0 \propto [\rho(T=0)]^{-1/x}$ where x is the power with which l_{ph} increases at $T \to 0, l_{ph} \propto T^{-x}$. Note that when this effect just starts, l_{ph} will very quickly decrease below L_0, where this picture should not be valid. However, for sufficiently disordered metals or alloys, a significant range of negative $d\rho/dT$ should exist. These observations are in qualitative agreement with Mooij's correlation[2] and with the results[1] on highly damaged $A15$'s.

As a point of principle, we emphasize again that the same behavior should obtain for $l_{ph} < \xi$ on the *localized* side. Thus, to distinguish between real localization and incipient effects, of the sort suggested here, one needs $l_{ph} > \xi$. This point is in agreement, in the localized phase, with Thouless.[12] It is remarkable, though, that localization affects the physics on the extended phase once $l_{ph} < \xi$. For $l_{ph} \gg \xi$ (or, more physically, when the time between inelastic collisions, τ_{ph}, becomes much larger than the time for quantum diffusion[12] over a length ξ) the extended and the localized regimes should behave very differently. In the former, ρ is only weakly temperature dependent. In the latter the "phonon-assisted" diffusion appears like a random walk with a step ξ (= localization length) and time τ_{ph}. Thus, we exprect in the latter case, independent of dimension,

$$\sigma \propto e^2 (dn/dE) \xi^2 / \tau_{ph}, \qquad (8)$$

where dn/dE is the density of states. Equation (8) assumes that the typical phonon energy satisfies $kT \gtrsim dE/dN$ (at scale ξ), as otherwise exponential dependence should follow.[12] Thus, 2D logarithmic behavior, as found in Ref. 14 for 2D samples, is in the range $l_{ph} < \xi$.[13] As emphasized above, in this range the behaviors in the extended and localized regions are similar.

These considerations strictly apply for the case of noninteracting electrons. It is entirely possible that the problem of localization where further couplings[5,7] are important may be quite different. This may include electron-electron interactions and self-trapping polaron effects. The latter effect may well be relevant for the saturation of $\rho(T)$ in clean $A15$ compounds, a case not covered by the present note. To get more quantitative results on $\sigma(T)$, $\sigma(E)$ has to be averaged[3] over the Fermi distribution by use of the Kubo-Greenwood formula. Such calculations have been done by McMillan.[16]

Useful discussions with A. Aharony, J. Black, G. J. Dolan, M. Jonson, W. L. McMillan, G. Reiter, and M. Strongin are gratefully acknowledged. This research was supported in part by the U. S. Department of Energy under Contract No. EY-76-C-02-0016.

---

[a] Permanent address.

[1] Z. Fisk and A. C. Lawson, Solid State Commun. 13, 277 (1973); M. Gurvitch, Ph.D. thesis, State University of New York, Stony Brook, 1978 (unpublished); M. Gurvitch, A. Ghosh, and M. Strongin, to be published.

[2] J. H. Mooij, Phys. Status Solidi A 17, 521 (1973).

[3] J. Woods Halley and Y. Imry, to be published.

[4] A. F. Ioffe and A. R. Regel, Prog. Semicond. 4, 237 (1960).

[5] N. F. Mott, *Metal Insulator Transitions* (Taylor and Francis, London, 1974).

[6] P. W. Anderson, Phys. Rev. 109, 1492 (1958).

[7] D. J. Thouless, Phys. Rep. 13, 93 (1974).

[8] M. Jonson and S. M. Girvin, Phys. Rev. Lett. 43, 1447 (1979).

[9] F. J. Wegner, Z. Phys. B25, 327 (1976).

[10] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).

[11] S.-k. Ma, unpublished note (1974); A. Nitzan, K. H. Freed, and M. H. Cohen, Phys. Rev. B 15, 4476 (1977); A. Aharony and Y. Imry, J. Phys. C 10, L487 (1977); H. G. Schuster, Z. Phys. B 31, 99 (1978). See, however, F. J. Wegner, to be published.

[12] D. J. Thouless, Phys. Rev. Lett. 39, 1167 (1977). By l_{ph} we mean the average length that the electron diffuses between inelastic collisions.

[13] P. W. Anderson, E. Abrahams, and T. V. Ramakrishnan, Phys. Rev. Lett. 43, 718 (1979).

[14] G. J. Dolan and D. D. Osheroff, Phys. Rev. Lett. 43, 721, 1690(E) (1979).

[15] P. A. Lee, Phys. Rev. Lett. 42, 1492 (1979); E. Domany and S. Sarker, to be published; D. C. Licciardello and D. J. Thouless, J. Phys. C 11, 925 (1978).

[16] W. L. McMillan, unpublished.