Bifurcations and Strange Behavior in Instability Saturation by Nonlinear Mode Coupling

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(Received 15 June 1979)

A simple model for the nonlinear saturation of an instability is studied via numerical solution of the resonant three-wave coupling equations in the case where the high-frequency wave is linearly unstable and the two lower-frequency waves are linearly damped. The solutions are shown to undergo interesting qualitative cahnges as the damping rate of the stable waves is increased. These changes include bifurcations to increasingly complicated periodic motions and the appearance of apparently chaotic motions, possibly indicative of the presence of a strange attractor.

An important general problem of plasma physics is that of determining the nonlinear state resulting from a linear instability. One of the elementary processes by which instability saturation can occur is resonant three-wave coupling of energy in the linearly unstable spectrum to waves which are linearly damped. Perhaps the simplest model of this process is embodied in the following normalized equations for a single interacting triplet of waves (with complex amplitudes C_1, C_2, C_3),

$$\dot{C}_1 = C_1 + C_2 C_3 \exp(i\delta t),$$

$$\dot{C}_{2,3} = -\gamma_{2,3} C_{2,3} - C_1 C_{3,2}^* \exp(-i\delta t),$$

where times have been normalized to the growth rate of wave 1 (the higher-frequency wave), and amplitudes are normalized to make the coefficient of the nonlinear term unity. $\gamma_{2,3}$ are the linear damping rates of waves $C_{2,3}$, and δ is the frequency mismatch. While these equations appear to be particularly simple, they will be shown to exhibit a surprisingly complex range of characteristic behavior which has not been previously examined. In particular, as parameters are varied, bifurcations to successibely higher-order periodic orbits are obtained, followed by solutions which are apparently chaotic. These apparently chaotic solutions have characteristics which suggest the presence of a strange attractor.[1] (Strange attractors have also recently been invoked in connection with low-Reynolds-number turbulence in fluids.[2])

Introducing $a_1 \exp(i\varphi_1) = C_1$, $a_{2,3}\exp(i\varphi_{2,3}) = C_{2,3} \times \exp(\frac{1}{2} i\delta t)$, and $\varphi = \varphi_1 - \varphi_2 - \varphi_3$, one may reduce the previous equations to four real first-order equations in the four real unknowns $a_{1,2,3}$ and φ. From these equations one readily obtains $d(a_2^2 - a_3^2)/dt = -2(\gamma_2 a_2^2 - \gamma_3 a_3^2)$. Thus, in the special case $\gamma_2 = \gamma_3 \equiv \gamma$, it is seen that $a_2^2 - a_3^2$ decreases exponentially in time. Furthermore, if $a_2 = a_3$ initially, then $a_2 = a_3$ for all subsequent times. Henceforth, we restrict our considerations to this case. The basic equations then become[3]

$$\dot{a}_1 = a_1 + a_2^2 \cos\varphi, \tag{1a}$$

$$\dot{a}_2 = -a_2(\gamma + a_1 \cos\varphi), \tag{1b}$$

$$\dot{\varphi} = -\delta + a_1^{-1}(2a_1^2 - a_2^2)\sin\varphi. \tag{1c}$$

Note that the system (1) depends on only two dimensionless parameters, δ and γ. We now proceed to a description of a digital-computer study of Eqs. (1) (analytical results will be reported elsewhere). For definiteness we set $\delta = 2$ and examine the range $\gamma = 1$ to $\gamma = 25$.

For all values of γ in this range, a_1, a_2, and φ have a temporally oscillatory character. The phase $\varphi(t)$ remained in the range $0 < \varphi(t) < \pi$ when initial conditions for φ were taken in this domain. To study the important characteristics of the motion of the system we shall use as one of our tools, surface of section plots: Each time the system orbit crosses $\varphi(t) = \frac{1}{2}\pi$ with $\dot{\varphi} < 0$, we plot its coordinates in the (a_1, a_2) plane. For all cases studied the orbit was apparently very strongly attracted to an arc in the surface of section, with the result that even the first piercing of the surface usually appeared to lie on this arc. Several important qualitative changes occur as γ is increased, and we now proceed to their description. For $1 < \gamma < 3$, $a_1(t)$ and $a_2(t)$ have an exponentially increasing envelope, and the system evolution is apparently unbounded because of the small damping rate γ (computations were terminated if either a_1 or a_2 exceeded 10^2). For $3 \leq \gamma \leq 8.5$, all initial conditions used lead to orbits which asymptote to a simple limit cycle. These features are illustrated in Fig. 1(a) and Figs. 2(a) and 2(b). For Figs. 2(a) and 2(b), notice the difference in the transient for $\gamma = 3$ and $\gamma = 8.5$. For $\gamma = 3$, the points in the surface of section converge to a single fixed point from one side. In contrast, for $\gamma = 8.5$, the points alternate se-

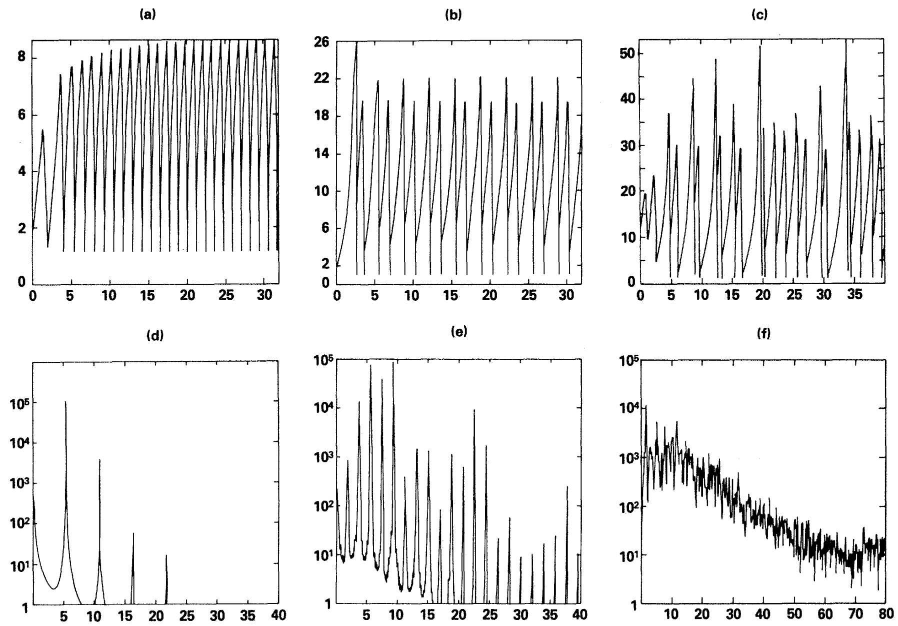

FIG. 1. Amplitude a_1 vs time for $\delta=2$ and (a) $\gamma=3$, (b) $\gamma=9$, and (c) $\gamma=15$; power spectrum of a_2 for $\delta=2$ and (d) $\gamma=3$, (e) $\gamma=9$, and (f) $\gamma=15$.

quentially about the fixed point while converging to it slowly. For γ somewhat larger than 8.5, the fixed point has become unstable and the points in the surface of section converge toward two points at which they sequentially appear. A bifurcation from a stable fixed point to an unstable fixed point and a stable two-point periodic cycle has occurred. This is illustrated in Fig. 2(c) with the corresponding time series shown in Fig. 1(b). As γ is increased, two periodic points are no longer reached from one side but by alternating on both sides [in analogy with Fig. 2(b)]. This is followed by the appearance of a four-point periodic cycle as the two-point cycle becomes unstable for $\gamma \gtrsim 11.9$ [see Fig. 2(d)]. The same procedure repeats for the four-point cycle which bifurcates into a stable eight-point periodic cycle for $\gamma \simeq 12.8$. For $\gamma \cong 13.15$, we observed a stable sixteen-point periodic cycle. For $\gamma \cong 13.16$, a thirty-two point cycle is observed. To summarize, we have observed, for $3<\gamma \lesssim 13.16$, a succession of bifurcations from a simple limit cycle, represented by a fixed point in the surface of section, to more-complicated limit cycles, characterized by a fixed points of periodicity 2^n after n bifurcations, and we have been able to distinguish n as large as 5. We also note that the range of values of γ for which a given periodicity is stable decreases with increasing n. For several values of γ in the range $13.4 \lesssim \gamma \lesssim 16.8$, the surface of section mappings with up to 1800 points show no periodicity at all. The sequence of points appears in an apparently random fashion and lies on an arc [Fig. 2(e)]. The time functions also appear chaotic [Fig. 1(c)]. Frequency power spectra of $a_2(t)$ in this case are broad [Fig. 1(f)] in contrast to spectra for the periodic case [Figs. 1(d) and 1(e)].

Increasing $\gamma \gtrsim 16.8$ we observe the rather sudden appearance of a stable three-point periodic cycle [Fig. 2(f)]. For $\gamma = 17.4$, the three-point cycle is unstable, and a bifurcation into a six-point cycle occurs. For $\gamma \cong 18.5$, apparently chaotic behavior reappears. Several values of γ in the range $18.5 \lesssim \gamma < 25$ were tried. All indicate chaotic motion along an arc in the surface of section.

Since the arc along which the points appear in the surface of section has no visible thickness we can reduce the surface of section to a one-dimensional mapping. To illustrate this, in Fig. 3 we have plotted $x_{n+1} \equiv a_2(t_{n+1})$ vs $x_n \equiv a_2(t_n)$ for $\gamma=15$, where t_n is the nth time at which the system orbit pierces the surface of section. The points so generated lie along a curve $x_{n+1}=F(x_n)$, which defines a "one-dimensional mapping" of the internal $7.6 \lesssim x_n \lesssim 38.5$ into itself. Note that the differential equations (1) are obviously in-

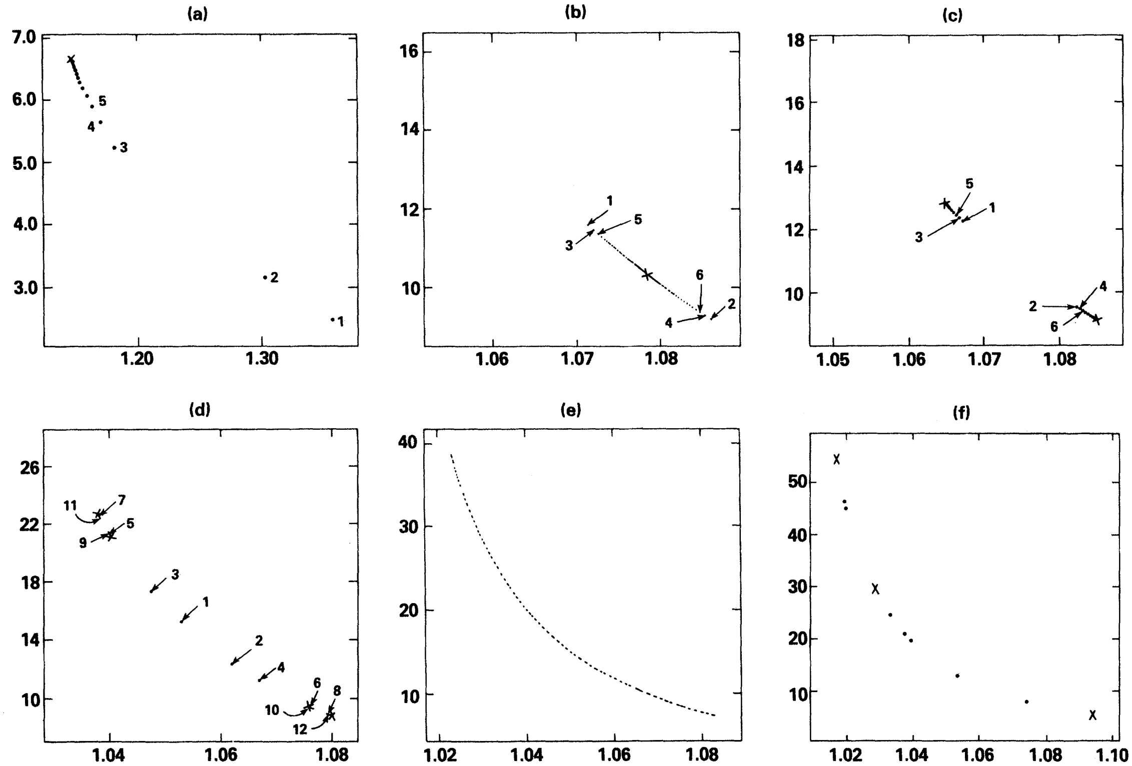

FIG. 2. Surface of section plots for $\delta = 2$ and (a) $\gamma = 3$, (b) $\gamma = 8.5$, (c) $\gamma = 9$, (d) $\gamma = 12$, (e) $\gamma = 15$, and (f) $\gamma = 17.2$. Numbers in the figures specify the order of appearance and crosses represent the stable periodic cycle.

vertible in time. Since the map in Fig. 3 is not invertible (some values of x_{n+1} correspond to *two* values of x_n), it cannot be an exact representation of the surface of section. Thus, the attracting set in the surface of section must actually have some thickness.

Recently, such one-dimensional maps have received considerable study.[4,5] For example, the simple mapping $x_{n+1} = Rx_n(1 - x_n)$ described in the review article by May[5] has a fixed point which

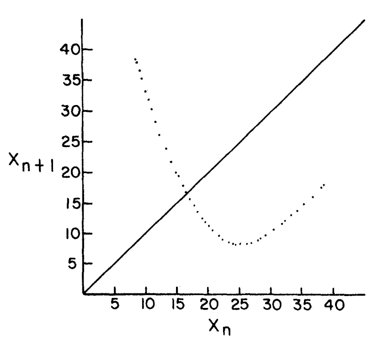

FIG. 3. Reduced one-dimensional mapping for $\delta = 2$, $\gamma = 15$.

bifurcates into a period two cycle at $R = 2$ and, as R is increased further, into cycles of period $4, 8, \ldots, 2^n, \ldots$ until an accumulation point occurs at $R \cong 3.57$. For $R \gtrsim 3.57$ the generated sequence can appear random. A stable three-point periodic cycle also appears in May's case at $R \cong 3.83$.

We have studied the one-dimensional mapping $x_{n+1} = F(x_n)$ obtained from the solutions of (1) in detail, and have found that almost all of the results which we have described can be obtained by a study of the changes in F as the decay rate γ is varied.

Similar arguments have been applied to the nonlinear equations of Lorenz which describe the Bénard instability.[1,6,7] In the case of the Lorenz system, the structure of the attractor has been extensively studied. These studies indicate that a strange attractor is probably present[1,7] and we expect that a similar result should apply in our case.

In conclusion, we have found that, as parameters are vaired, Eqs. (1) exhibit sequences of bifurcations as well as chaotic behavior characteristic of a strange attractor. It is anticipated that characteristic behavior similar to that found here will also occur in a variety of other problems in plasma physics.[8]

We thank O. Manley, Y. Trève, P. Holmes, and J. Greene for helpful discussion. This work was supported by the U. S. Department of Energy.

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[3]Computer analysis and discussion of this system have also been presented by S. Ya. Vyshkind and M. I. Rabinovich [Zh. Eksp. Teor. Fiz. 71, 557 (1976) [Sov. Phys. JETP 44, 292 (1976)]. Their results, however, are quite different from ours and are contradicted by some simple analytical results which can be obtained from Eqs. (1).

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[5]R. M. May, Nature 261, 459 (1976).

[6]E. N. Lorenz, J. Atmos. Sci. 20, 130 (1963).

[7]O. E. Landord, III, in *Statistical Mechanics and Dynamical Systems* (Mathematics Department, Duke University, Durham, N. C., 1976), Chap. IV.

[8]In this connection some preliminary results of J. M. Greene (private communication) on the saturation of the dissipative trapped-ion mode are relevant.

Raman Spectrum of Solid Orthodeuterium to 150 kbar at 5 K

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(Received 31 December 1979)

We have studied samples of 98.5% o-D_2 in a diamond anvil cell to pressures of 150 kbar at 5 K by means of Raman scattering. At 50–60 kbar the E_{2g} phonon and roton cross and hybridize. The coupling has been determined. The phase transition from the symmetric ground state at low pressure to an orientationally ordered ground state, predicted to occur between 31 and 73 kbar, does not take place. A predicted hcp-fcc structural phase transition is not observed.

In this Letter we present our results for the first low-temperature, ultrahigh-pressure measurements on one of the hydrogen isotopes, namely deuterium. Sharma, Mao, and Bell[1] have recently pressurized H_2 to 630 kbar (63 GPa); however, their work was at room temperature on normal hydrogen. Here we demonstrate the importance of low-temperature studies on a pure ortho-para species for obtaining detailed information of the properties of the solid molecular hydrogens (H_2,D_2, etc.). Our measurements provide new and unexpected results for the interactions, the excitation spectrum, and the structure of solid molecular deuterium.

At zero pressure the molecules in solid parahydrogen (p-H_2) and orthodeuterium (o-D_2) are in the spherically symmetric $J=0$ rotational state and the lattice is hcp (space group $I_{6h}{}^4$). The low-lying lattice excitations are phonons and $J=2$ rotons.[2] As the density is increased, anisotropic interactions lead to mixing of the higher rotational states into the ground state.[3] At a sufficiently high density the mixing of $J=2$ into the $J=0$ single-molecule states becomes so severe that the symmetry of the molecules will be broken and the ground state will be orientationally ordered.[4,5] In the broken-symmetry phase the excitations will be librons[6,7] and phonons with a characteristic spectrum. This phase has been predicted to occur at pressures of 31 (Ref. 5) and 73 (Ref. 4) kbar for o-D_2 and 86 (Ref. 5) and 270 (Ref. 4) kbar for p-H_2.[8] The differences in the predictions of the two theories evidently arise from the use of different anisotropic-potential parameters. From our Raman spectra we see no indication of the broken-symmetry phase transition up to 150 kbar. Since theories[7] which go beyond the mean-field theories of Refs. 4 and 5 do not predict substantially different critical densities for the same potentials, we interpret this to mean that the anisotropic interactions have a weaker radial dependence than had been thought

