ficult to evaluate the second-order Born amplitude with the Coulomb Green's function, this latter procedure being fully consistent.

Cross sections for electron capture from hydrogenlike atoms of high atomic number have been calculated by Olson[7] using a classical trajectory Monte-Carlo method. Newton's equations were solved essentially exactly, and not perturbatively, so that the only error lies in the classical approximation.

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Turbulence near Onset of Convection

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New long-term (days) measurements of the evolution of turbulence for a Rayleigh-Bénard system with aspect ratio $\Gamma = 4.72$ reveal a turbulent state, with a threshold near the critical Rayleigh number R_c, which consists of a random background time dependence and rare, randomly spaced, major events. These events are discussed in terms of the analog of a particle under the influence of a stochastic driving force and in a potential with two minima.

It was reported previously[1,2] that the sequence of events leading to nonperiodic, or turbulent, flow in a horizontal layer of fluid heated from below (Rayleigh-Bénard system) is qualitatively altered by changing the aspect ratio $\Gamma \equiv L/d$ of the sample (L is the radius and d is the height of the cylindrical container). Particularly surprising was the observation that nonperiodic behavior occurred at Rayleigh numbers R immediately above the critical Rayleigh number R_c for the onset of flow when Γ was large ($\Gamma = 57$). This experimental result is difficult to reconcile with the stability analysis of Schlüter, Lortz, and Busse[3] who predicted on the basis of the deterministic equations of motion of the laterally infinite system that there should be a range of R above R_c over which a fluid velocity field consisting of a time-independent system of rolls will be stable. At an intermediate value of Γ ($\Gamma = 4.72$), experiments over time periods of many hours had led us to believe that the fluid flow was time independent for $R \lesssim 2R_c$, and had demonstrated that the system was *obviously* turbulent for greater R.[1,2,4] This result also differs from the analysis of the deterministic equations of motion for the laterally infinite system which predicts[5] that the first instability of the steady convection should be to a periodic state and should occur only when $R \gtrsim 5R_c$. Indeed, the experiment revealed that structure in the broadband spectrum evolves for $R \geq 4.7R_c$ at frequencies which are consistent with the predicted periodic state[1,2]; but the bifurcation which yields this state occurs only *after* the system is already turbulent. In the present Letter we report new results which were obtained for the medium-aspect-ratio system ($\Gamma = 4.72$) on a time scale of many days rather than many hours. They show that rare, randomly spaced, major events occur for R well below $2R_c$ where we previously thought the fluid flow to be stationary. The new results indicate that the onset of chaotic behavior occurs very close to R_c even for medium-aspect-ratio systems, and the data suggest that the time scale of the turbulence diverges exponentially as $R - R_c$ vanishes. We will discuss the observations in terms of a model with an external stochastic driving force.

The apparatus, fluid, temperature, and general experimental procedure for this investigation were the same as those used previously[1] (cell A). In this experiment, the Rayleigh number was increased continuously and slowly $[(t_v/R_c)(dR/dt)\approx 10^{-2}$, vertical diffusion time $t_v = d^2/\kappa$, κ = thermal diffusivity$]$ to a value above R_c, but well below $2R_c$ where previous work had shown the system to become *obviously* turbulent. The heat current q was then held constant for many days, and the temperature response was measured. We expect, of course, that any time dependence of the temperature difference across the cell is caused by a time dependence of the fluid velocity field. The temperature difference, expressed as the reduced Rayleigh number R/R_c, is shown in Fig. 1 for a particular value of q which was held constant for about 10 days. The duration of this run corresponds to approximately 3×10^3 vertical thermal diffusion times since $t_v = 311$ sec. The data reveal a very slow, nonperiodic background time dependence of the flow, with rare major events occurring in addition at random time intervals. The major events, although they appear sharp on the scale of Fig. 1, have a width of $5t_v$ to $10t_v$. Their amplitudes are remarkably uniform, and they are always excursions to larger R (smaller heat transport). They occur only following a substantial rise in R/R_c due to the random background motion. The rms amplitude of the nonperiodic background is only 0.002, but this is a factor of about 4 larger than long-term experimental drift and noise.

Spectral analysis of the nonperiodic background yielded a broad power spectrum which had a maximum at zero frequency and a first moment about equal to $4\times 10^{-3}/t_v$ ($\approx 1\times 10^{-5}$ Hz). At high frequency, the spectrum varied as f^{-m}, with $m = 2.2\pm 0.4$.[6]

We look upon the nonperiodic background time dependence as the result of random motion of the system in the vicinity of a minimum of a potential, provoked possibly by random external forces.[7] The major events can be thought of as well-defined transitions to a second minimum[8] which are rare because an activation energy Δ_g considerably larger than the strength of the driving force separates the two minima. The random occurrence and rather uniform heights of the events, and their occurrence only when the random background motion already has increased R/R_c considerably, are consistent with this picture. Their relatively short lifetimes, which are comparable to the transit time between the two minima of the potential, suggests that the second minimum is separated from the first by an activation energy Δ_0 which is of the same order or smaller than the driving noise strength.

A number of runs at different q were performed, but usually they were only of 4 or 5 days duration. In each case, q was changed from its previous value without returning to $q = 0$. The results are summarized in Fig. 2. The data sets are labeled $a, b, c, \ldots$ to indicate the sequence in which they were taken. Run d is a section of the data in Fig. 1 which is typical of the runs with $R/R_c \lesssim 1.7$.

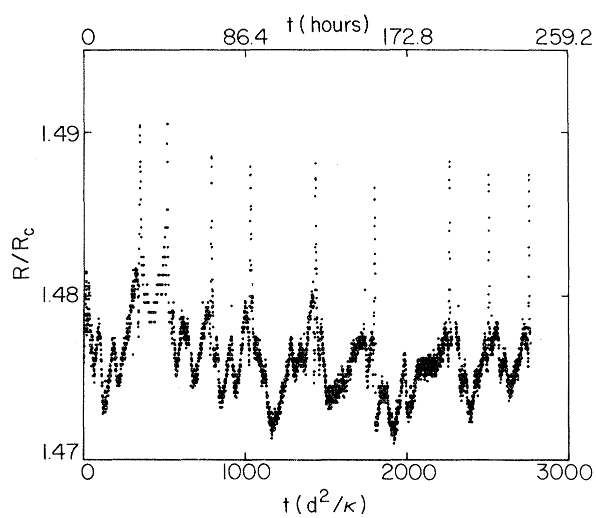

FIG. 1. The Rayleigh number as a function of time at constant heat current q.

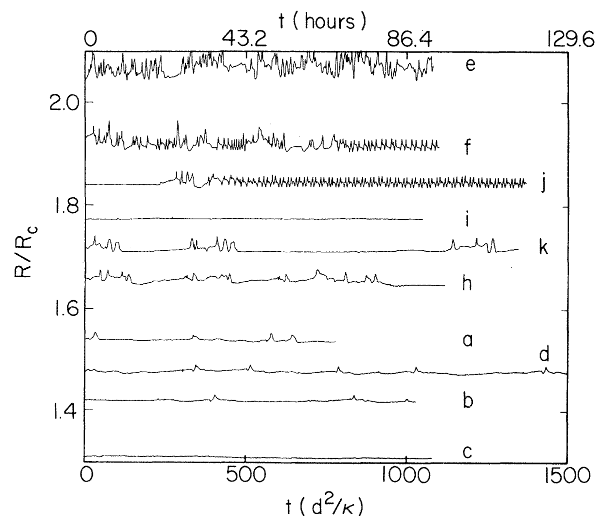

FIG. 2. The Rayleigh number as a function of time at several constant values of the heat current.

Run c shows no major events, but its noise level is still above the experimental noise level. It is apparent that the frequency of events increases with increasing R/R_c. The number of events n per unit time t_v is shown on a logarithmic scale as a function of ϵ^{-1}, $\epsilon \equiv R/R_c - 1$, in Fig. 3 as solid circles. The data can be represented by the equation

$$n = n_0 \exp(-\Delta_0/\epsilon), \quad (1)$$

with $n_0 \cong 1.0$ and $\Delta_0 \cong 2.7$. Within the picture described above, Eq. (1) implies either that the activation energy Δ_g diverges at $\epsilon = 0$ as ϵ^{-1}, or that the system provides an external-noise amplification which is proportional to ϵ. On the basis of the data, we cannot rule out, however, that the argument of the exponential in Eq. (1) is $-\Delta_0/(\epsilon - \epsilon_0)$ with $\epsilon_0 \lesssim 0.15$.

Returning to Fig. 2, it is clear that runs i and j are qualitatively different from those at smaller R. Run i shows no events, and its noise level is close to the long-term experimental noise level. As far as we can tell, this state corresponds to a fixed point, or stationary fluid flow. Upon going from i to j, the stationary state, although long lived, becomes unstable and decays to a limit cycle. The spectrum of the second half of j is periodic, with instrumentally sharp lines. The limit cycle was obtained also (run f) after coming from a large R turbulent state (e). In this case, the turbulent state persisted for over $700 t_v$ before the limit cycle was finally fully established. Not shown in Fig. 2 is run g, which corresponded to the same q as i.

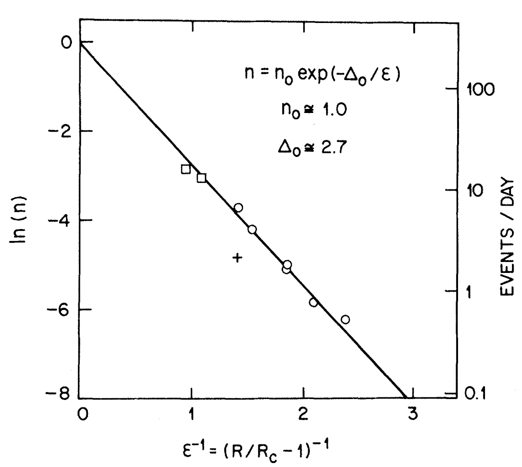

FIG. 3. The number of major turbulent events per vertical thermal diffusion time as a function of $(R/R_c - 1)^{-1}$.

In that case the limit cycle established in f was unstable with respect to the fixed point, but the fixed point was not reached until $t \cong 500 t_v$. The existence of long-lived unstable orbits is interesting because it has been observed below the onset of turbulence for the Lorenz model.[9] Another interesting case represented in Fig. 2 is run k. Here the system switches back and forth between the nonperiodic state which prevails at smaller R and the fixed point. During the two long quiet periods, the noise decays to the instrumental level. If the number of events n is calculated on the basis of the entire duration of the run, the cross in Fig. 3 is obtained. This is clearly too low compared to the other data. If the two long quiet periods are not included in the time interval, then the solid circle at $\epsilon^{-1} = 1.4$ is obtained and the result for n is consistent with the other data. Run e represents the obviously turbulent state which was reported previously.[1,2,4] We cannot be sure that this state is basically the same as the nonperiodic state at small R, but there is some evidence suggesting that it is. Major positive excursions of R are so frequent in run e that they overlap and are difficult to count. Nonetheless, estimates of n from run e and the beginning of run f are shown in Fig. 3 as solid squares and are consistent with the line drawn through the other data. Run e has a skewness which is significantly greater than zero (of order 0.5). Previous measurements with $2R_c \lesssim R \lesssim 3R_c$ had yielded results with a nonzero skewness[1,4] (as large as 1.0 near $2.5R_c$). This is consistent with a random motion of the system in an asymmetric potential. At even larger R, the skewness decreases again and approaches a value of 0.2 ± 0.1. For $R \gtrsim 2R_c$, the power spectrum of R has been discussed elsewhere.[1,4] It has a maximum at zero frequency, but the power decreases at large f as f^{-m} with $m = 4.0 \pm 0.2$.[6]

Since the system with $\Gamma = 57$ became obviously turbulent very close to R_c,[1,2] we expect that Δ_0 decreases with increasing Γ. This is also consistent with the absence of time-dependent flow for $\Gamma = 2$ and $R \lesssim 7R_c$.[1,2] In this latter small-Γ case, we presume that the stochastically generated nonperiodic flow occurs noticeably only at such high Rayleigh numbers that the sequence of bifurcations leading to turbulence from the deterministic equations of motion could precede it. Note that in this case the spectrum of the turbulent state is qualitatively different,[1,2] with the power concentrated at finite frequencies,

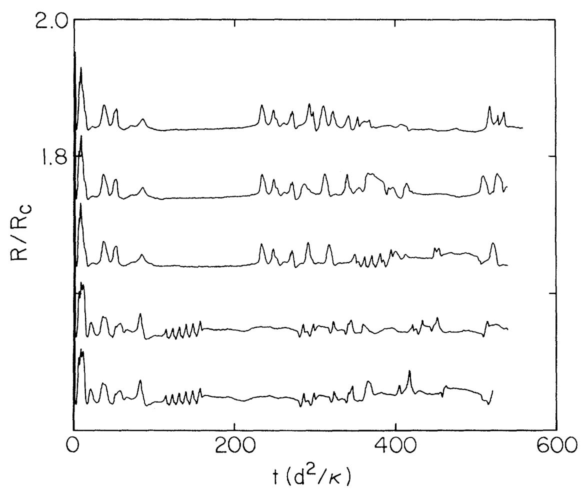

FIG. 4. The Rayleigh number as a function of time after the heat current q was switched discontinuously from zero to a constant finite value. In each case, the same value of q was used. The top data set is placed in its proper position on the ordinate scale, and the other sets have been displaced downwards by successive increments of 0.1 in R/R_c.

suggesting that a completely different mechanism is responsible for the nonperiodic behavior.

Finally, we would like to mention a different experiment which may have a bearing upon the same general problem. The results reported above were all obtained with initial conditions close to those of the final state. We also made measurements in which the initial conditions were *far* from those of the final state, by switching q discontinuously from zero to some finite value. The results of five nominally identical runs are shown in Fig. 4. Although no two nominally identical experiments can be identical in a rigorous sense, we expect that all the experimental runs start in a very small but finite neighborhood of phase space. We see that the fluid motion evolves in a very complicated manner, yielding many excursions of apparently random shape in R. The remarkable result of these measurements is that $R(t)$ is identical within experimental noise for the top three runs up to the very large time $t \cong 270t_v$ (about 1 day). The bottom two runs represent a different time evolution, but agree with each other until $t \cong 360t_v$. Since we do not expect to be able to reproduce initial conditions *exactly,* we conclude from the reproducibility of the data over such long time intervals that there is only a small number of very complicated but *discrete* orbits available to the system in the nonvanishing region of phase phace corresponding to the initial conditions. Eventually, these discrete orbits may come close to other orbits (near $t = 270t_v$, for instance) and stochastic forces can cause a random selection of the subsequent history of the experiment.

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